Intensity Models for Credit Risk

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Contents

Abstract

In this thesis, we examine the use of intensity models in modelling credit risk. In this approach, we assume the existence of a stochastic process λ_t representing the instantaneous default probability of an obligor. We then specify various Lévy processes for the intensity process λ_t, with particular emphasis on non-Gaussian Ornstein-Uhlenbeck process, derive various theoretical properties of these models, and calibrate these models to credit market data. We find that Gamma-OU and Inverse Gaussian-OU Lévy processes can be effectively applied in an intensity context, and provide several advantages over conventional Poisson processes.

We then extend the intensity approach to modelling multivariate credit derivatives. We discuss various proposed multivariate intensity models, and extend several copula models using our candidate Lévy processes to derive our marginal distribution. Our empirical results are then compared and contrasted with previous results in the literature. Again, we show that non-Gaussian OU processes offer plausible correlation sensitivities and VaR profiles for several multivariate credit derivatives.

Acknowledgments

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Chapter 1

An Introduction to Credit Risk Modelling

1.1. Credit Derivatives

Since the global financial crisis of 2008, much attention has been placed on financial products known as *credit derivatives*, to which some blame for the crisis has been attributed. Credit derivatives were first introduced in 1994 as a product to hedge *credit risk* - broadly speaking, the risk that an obligor does not honour his payment obligations.

Credit risk is inherent in all financial markets. For example, a bank takes on credit risk when it extends a loan to a person to buy a house - it is exposed to the person *defaulting* on his debt. With credit derivatives, market participants are able to hedge, transfer and manage their exposure to credit risk - it can be regarded as an insurance against default. A credit derivative is a financial derivative whose payoff is affected by the default of a *reference entity* (or a basket of reference entities).

The cornerstone of the credit derivatives market is the *credit default swap* (CDS). In a CDS, a protection buyer transfers default risk to a protection seller for a predetermined amount of time T. The protection buyer makes predetermined payments to the seller, until the maturity date T of the contract, or a default event, whichever occurs first. If a default event occurs, the protection seller pays the protection buyer the notional amount of the contract [58]. Credit default swaps are the most commonly traded credit derivative, accounting for approximately 30% of the market. The Bank of International Settlements (BIS) recently reported the notional amount of outstanding CDS contracts exceeds $30 trillion. Figure 1.1 illustrates the rapid growth in the CDS market leading up to the financial crisis of 2008.

Credit default swaps are traded on almost all forms of debt by a variety of market participants. Typically, protection buyers are commercial banks and pension funds seeking to hedge their credit risk, while protection sellers are often hedge funds in a speculative role. Credit default swaps are traded on nearly all forms of debt, from CCC sovereign issuers (such as Greece) to AAA corporations (such as Microsoft). The credit default swap market is highly active, with over $13 trillion in net single-name notional exposure being traded in the week of 1/10/11 [37].

A commonly traded derivative of the credit default swap is a credit default swap index, which allows market participants to buy and sell credit protection on a basket of reference entities. Common CDS indices include the CDX (125 investment-grade North American credits), iTraxx Europe (125 investment-grade European credits), and the CDX HY (100 high yield North American credits). Over $6 trillion in net CDS index exposure was traded in the week of 1/10/11 [37].

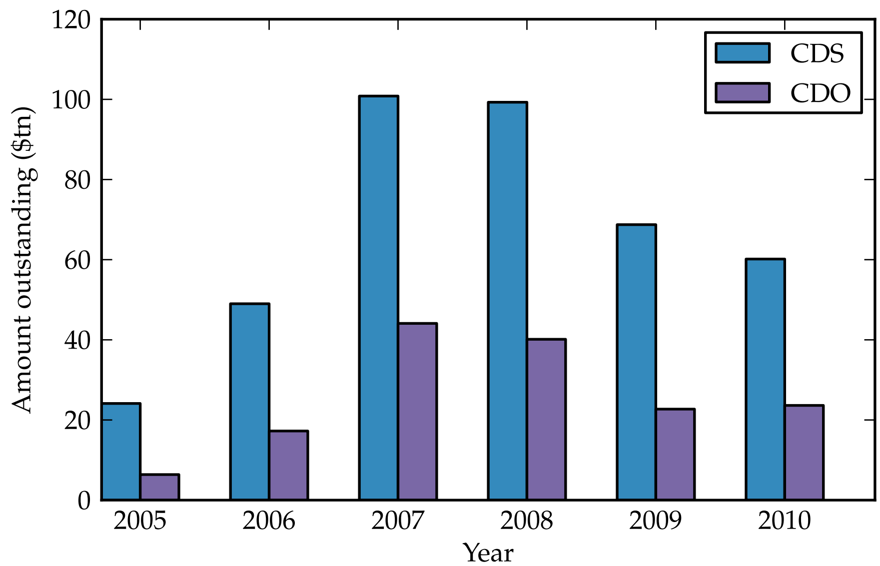

Figure 1.1. Global credit default swaps and collateralised debt obligations notional outstanding, in trillions of US dollars. *Source:* Bank of International Settlements.

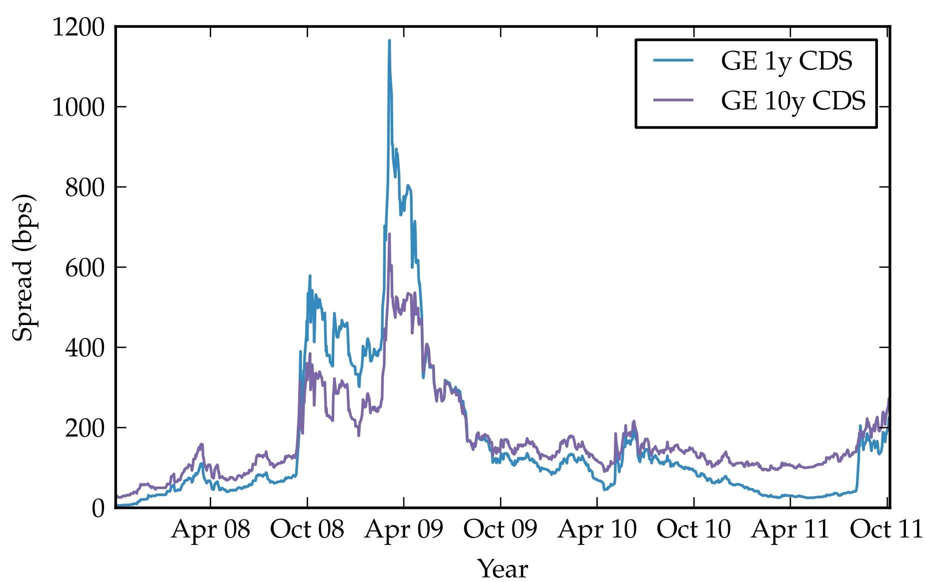

Figure 1.2. Performance of General Electric CDS spreads from 5/10/2007 to 6/10/2011. Higher spreads indicate a riskier credit environment. *Source:* Goldman Sachs.

1.2. Credit Risk Modelling

Credit markets have experienced exponential growth in recent years, and consequently regulatory bodies have acted to ensure financial institutions account for increased credit exposure. Much research has gone into developing models to assess credit risk. These models as typically classified into two categories: *structural* models and *intensity* models.

Structural models seek to model the value of the assets on which the debt is written. These models were inspired by the classical Black-Scholes insight that the debt of a firm can be viewed as a contingent claim on the assets of the firm [11, 48]. Typically, the dynamics of the asset value of a firm V_t is given and the event of a default is specified in terms of boundary conditions on this process [10].

A large number of variations on the firm-value approach have been developed over the past forty years, with varying results. The analysis of [41] and [64], amongst others, found that early firm-value models offered little explanatory power compared to more naïve procedures.

In a response to these issues, researchers turned to so called *intensity* models, also known as *hazard-rate* or *reduced-form* models, that seek to directly model the instantaneous probability of default [58]. The key distinction between structural models and intensity models are in the level of knowledge assumed by the modeller. Indeed,

> Structural models assume that the modeller has the same information set as the firm's manager - complete knowledge of all the firm's assets and liabilities. In most situations, this knowledge leads to a predictable default time. In contrast, reduced form models assume that the modeller has the same information set as the market - incomplete knowledge of the firms condition. In most cases, this imperfect knowledge leads to an inaccessible default time. And so we argue that the key distinction between structural and reduced-form models is not whether the default time is predictable or inaccessible, but whether or not the information set is observed by the market. Consequently, for pricing and hedging, reduced-form models are the preferred methodology. [38]

This approach, instigated in [39], assumes that at any instant time there is a probability that a default will occur. Default is then defined as the first jump of a counting process N_t, where jumps occur with at an intensity rate λ_t. The behaviour of intensity over time corresponds to the *health* of the reference entity. Intensity models assume that default is not induced by market observables or fundamentals, but is an exogenous component independent of the default-free information [13].

1.3. Modelling Assumptions

We will be working in a *filtered probability space* $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F})$ equipped with a default-free filtration $\mathbb{F}$. A filtration is a non-decreasing family $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ of sub-σ-algebras of $\mathcal{F}$. The filtration represents the information flow of interest rates, intensities and possibly other default-free market quantities [13].

On occasion, we will deal with an augmented filtration $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$, where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau < u\}, u \leq t),$$

representing the flow of information on whether the default time τ of a reference entity occurred before t, augmented with the usual default-free information $\mathcal{F}_t$.

We assume the existence of a short-rate process r_t, adapted to the filtration $\mathcal{F}_t$ with the property that, for any times s and $t > s$, an investment of one unit at time s, reinvested in a short-term savings account until time t will yield a market value of $e^{\int_s^t r(u)\,du}$. Throughout this thesis, we assume that the short rate r_t is uncorrelated with our default intensity λ_t.

We assume that our financial market model is arbitrage-free, in the sense that there exists an *equivalent martingale measure* $\mathbb{Q}$, such that the price process of any tradable security which pays no coupons or dividends, is a $\mathbb{F}$-martingale under $\mathbb{Q}$ when discounted by the savings account B, given as

$$B_t = e^{\int_0^t r_u\,du}$$

From martingale pricing theory, we recall that there may be infinitely many risk-neutral measures in our market. We assume that the tradable securities consist of both default-free bonds and credit default swaps (defined in §3.2). In this case, the equivalent martingale measure $\mathbb{P}$ to be used for pricing purposes is implied by the market prices [9]. The prices of traded credit default swaps can be used to imply the cumulative distribution function of the default time τ under $\mathbb{P}$.

Hereafter, we will exclusively use this market implied risk neutral measure. Expressions such as $\mathbb{P}(\tau > t)$ and $\mathbb{E}(V_T)$ are understood as with respect to this market-implied risk-neutral measure.

Given our risk neutral measure, we can calculate the *discount factor* from times t to T, the rate at which cash-flows at time T should be discounted to at time t. We calculate this discount factor $P(t, T)$ by the formula

$$P(t,T) = \mathbb{E}\left(\exp\left(-\int_t^T r_s\,ds\right) \mid \mathcal{F}_t\right), \tag{1.1}$$

given in [49].

1.4. Remarks on Notation

Throughout this thesis, we denote stochastic processes $\{X_t(\omega), 0 \leq t \leq T, \omega \in \Omega\}$ as simply X_t. Distributions will be denoted in small caps, for example - Poisson(λ).

Chapter 2

An Introduction to Lévy Processes

2.1. Lévy Processes in Finance

Lévy processes are the simplest class of processes whose paths consist of continuous motion interspersed with jump discontinuities of random size appearing at random times [3]. They can be viewed as a generalisation of random walks to continuous time. In recent years Lévy process have become increasing popular as modelling tools in mathematical finance, as results indicate Lévy processes can provide better fits to empirical data than geometric Brownian motion [12, 17, 31].

This thesis focuses on modelling credit spreads in an intensity framework, and Lévy processes with jumps will be used as models of the default intensity. The Lévy process framework is very attractive in this context for several reasons.

Primarily, the market standard for modelling credit spreads is a derivative of Black's model for interest rate markets [52], which assumes that the log-returns on credit spreads are normally distributed. Empirical investigations of credit spread returns, summarised in [58], suggest that log-returns follow a fat-tailed distribution, which suggest that alternative Lévy processes may be more suited to modelling credit risk.

Secondly, empirical investigations also showed that credit spreads exhibit significant mean-reversion and jump characteristics [53]. This gives further evidence in favour of using Lévy processes as models for the intensity rate.

2.2. Properties of Lévy Processes

Formally, we can define a Lévy process by the following four conditions.

Definition 2.1 (Lévy Process [3]). *A stochastic process X_t is a Lévy process if the following conditions hold:*

(i) $X_0 = 0$ *almost surely.*
(ii) X_t *has stationary increments.*
(iii) X_t *has independent increments.*
(iv) The map $t \mapsto X_t$ *is almost surely right continuous with left limits (càdlàg) .*

The first statement is a convenient normalisation, and can be replaced with any constant $c \in \mathbb{R}$.

As our framework will use Lévy processes as models for the intensity rate in our intensity models for credit risk, we will additionally require that our Lévy processes are non-negative.

A useful property satisfied by Lévy processes is that of *infinite divisibility*.

Definition 2.2 (Infinite Divisibility). *Let $\varphi(u)$ be the characteristic function of a random variable X. If, for every positive integer n, $\varphi(u)$ is also the nth power of a characteristic function, we say that the distribution is infinitely divisible.*

Thus, for any n, we can write

$$X = Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)},$$

where the $Y_i^{(n)}$ are independent and identically distributed random variables with characteristic function $(\varphi(u))^{1/n}$.

One of the most useful characteristics of a Lévy process X_t is that knowing the distribution of X_t at a specific time - say X_1 - completely determines the distribution at all time t. This is intuitively clear from the stationary and independent increments. The following theorem illustrates the relationship between Lévy processes and infinitely divisible distributions.

Theorem 2.3 (Infinite Divisibility of Lévy Processes (Proposition 1.3.1 of [4])). *Let X_t be a Lévy process. Then X_t has an infinitely divisible distribution F for every t.*

Conversely, if F is an infinitely divisible distribution there exists a Lévy process X_t such that the distribution of X_1 is given by F.

Proof. To show the forward implication, let X_t be a Lévy process and let $n \in \mathbb{N}$. Then by the stationary independent increments property, we have

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}). \tag{2.4}$$

Now, for all $u \in \mathbb{R}$ and $t \geq 0$, let

$$\psi_t(u) = \log \mathbb{E}\left(e^{iuX_t}\right)$$

Then by (2.4) we have for any two positive integers p, q that

$$p\psi_1(u) = \psi_p(u) = q\psi_{p/q}(u)$$

and thus for any rational $t > 0$,

$$\psi_t(u) = t\psi_1(u). \tag{2.5}$$

If t is irrational, then let $t_n \downarrow t$ be a decreasing sequence of rationals. Then by the almost sure right continuity of X_t, we have right continuity of $e^{-\psi_t(u)}$ and hence (2.5) holds for any $t \geq 0$. Thus we achieve our result that for all $t \geq 0$,

$$\mathbb{E}\left(e^{iuX_t}\right) = e^{t\psi(u)}$$

where $\psi(u) = \psi_1(u)$ is the characteristic exponent of X_1.

The reverse implication is proven in a similar manner. □

Corollary 2.6. *Let X_t be a Lévy process, and let the characteristic function of X_1 be $\varphi(u)$. Then the distribution of an increment over $[s, s+t]$, $s, t > 0$, has $(\varphi(u))^t$ as a distribution function.*

Proof. By the stationary increments property, we have that $X_{s+t} - X_s \sim X_t$. Then from Theorem 2.3, the characteristic function of X_t is equal to $(\varphi(u))^t$. □

The characteristic functions of infinitely divisible probability measures, and hence Lévy processes, were completely characterised by Lévy and Khintchine in the 1930s. The following result, given without proof, is a fundamental result in the study of Lévy processes.

Theorem 2.7 (Lévy-Khintchine formula (Theorem 1.2.14 of [4])). *Let X_t be an infinitely divisible process, and define the characteristic exponent $\psi(u)$ by*

$$\psi(u) = \log \mathbb{E}\left(e^{iuX_1}\right) = \log \varphi(u),$$

where $\varphi(u)$ is the characteristic function of X_1. Then the characteristic exponent $\psi(u)$ satisfies the following Lévy-Khintchine formula

$$\psi(u) = i\gamma u - \frac{\zeta^2}{2}u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\, 1_{|x|<1}\right)\, d\nu \tag{2.8}$$

where $\gamma \in \mathbb{R}$, $\zeta^2 \geq 0$, 1_A is the indicator function of A, and ν is a measure of $\mathbb{R}\backslash\{0\}$ such that

$$\int_{\mathbb{R}} \min(1, x^2)\, d\nu < \infty$$

We call the measure ν the Lévy measure, and it dictates how the jumps occur - jumps of sizes in the set A occur according to a Poisson process with parameters $\nu(A)$ [58].

We see that given parameters $\left(\gamma, \zeta^2, d\nu\right)$, we uniquely determine the corresponding Lévy process. Thus, we call these parameters $\left(\gamma, \zeta^2, d\nu\right)$ the *Lévy triplet*.

We finish our theoretical background with a beautiful result, due to Lévy and Itô, providing a decomposition of any Lévy process into drift, diffusion, and jump components.

Theorem 2.9 (The Lévy-Itô decomposition (Theorem 2.4.16 of [4])). *Let X_t be a Lévy process. Then we can decompose X_t as*

$$X_t = \beta t + \sigma B_t + J_t + M_t$$

where B_t is a Brownian motion, J_t is a compound Poisson process, and M_t is a square integrable pure jump martingale that almost surely has a countable number of jumps on a finite interval.

Proof. As X_t is a Lévy process, then by Theorem 2.7, we can characterise X_t by it's Lévy triplet $\left(\gamma, \zeta^2, d\nu\right)$.

Consider rewriting (2.8) as follows

$$\psi(u) = \underbrace{i\gamma u - \frac{\zeta^2}{2}u^2}_{(\star)}$$
$$+ \underbrace{\nu\left(\mathbb{R}\backslash(-1,1)\right)\int_{|x|\geq 1}\left(e^{iux}-1\right)\frac{d\nu}{\nu\left(\mathbb{R}\backslash(-1,1)\right)}}_{(\star\star)}$$
$$+ \underbrace{\int_{|x|<1}\left(e^{iux}-1-iux\right)\,d\nu}_{(\star\star\star)}$$

We show that $(\star)$ is the characteristic exponent of a Brownian motion with drift, $(\star\star)$ is the characteristic exponent of a Poisson point process, and $(\star\star\star)$ is the the characteristic exponent of a pure jump martingale that almost surely has a countable number of jumps on a finite interval.

Consider $(\star)$. Let $Y_t^{(1)} = \zeta B_t + \gamma t$ with ζ, γ constants and B_t a Brownian motion. Then we have that the characteristic exponent of $Y_t^{(1)}$ is given by

$$\begin{aligned}\log\mathbb{E}\left(e^{iuY_1^{(1)}}\right) &= \log\left(e^{iu\gamma - \frac{\zeta^2}{2}u^2}\right) \quad \text{by properties of the normal distribution}\\ &= iu\gamma - \frac{\zeta^2}{2}u^2\end{aligned}$$

which corresponds to our expression for $(\star)$.

Consider now $(\star\star)$. Let $Y_t^{(2)}$ be a compound Poisson process, that is,

$$Y_t^{(2)} = \sum_{i=1}^{N_t}\xi_i$$

with (ξ_i) a sequence of independent, identically distributed random variables with distribution function F and N_t a Poisson process with parameter λ (defined in §2.3.1), independent of the sequence (ξ_i). Then we can calculate the characteristic function of $Y_1^{(2)}$ by

$$\begin{aligned}\mathbb{E}\left(e^{iuY_1^{(2)}}\right) &= \mathbb{E}\left(e^{iu\sum_{i=1}^{N_1}\xi_i}\right)\\ &= \sum_{n=0}^{\infty}\mathbb{E}\left(e^{iu\sum_{i=1}^{n}\xi_i}\right)\frac{e^{-\lambda}\lambda^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\int_{\mathbb{R}}e^{iux}\,dF\right)^n\frac{e^{-\lambda}\lambda^n}{n!}\\ &= \exp\left(\lambda\int_{\mathbb{R}}\left(e^{iux}-1\right)\,dF\right).\end{aligned}$$

Thus the characteristic exponent of $Y_t^{(2)}$ is given by

$$\log \mathbb{E}\left(e^{iuY_1^{(2)}}\right) = \lambda \int_{\mathbb{R}} \left(e^{iux} - 1\right)\, dF$$

Thus setting $\lambda = \nu\left(\mathbb{R}\backslash(-1,1)\right)$ and letting ξ_i be distributed with distribution $\frac{d\nu}{\nu(\mathbb{R}\backslash(-1,1))}$ on $\{x \mid |x| \geq 1\}$, we obtain that the characteristic exponent of $Y_t^{(2)}$ corresponds with $(\star\star)$.[1]

Finally, consider $(\star\star\star)$. It can be shown (see Appendix A1) that there exists a square integrable pure jump martingale $Y_t^{(3)}$ with characteristic exponent $(\star\star\star)$.

Finally, letting $X_t = Y_t^{(1)} + Y_t^{(2)} + Y_t^{(3)}$, where the $Y^{(i)}$ are independent and defined above, we have that the characteristic exponent of X_t satisfies (2.8). This completes our proof of the proposition. □

2.3. Examples of Lévy Processes

Here we present several examples of Lévy processes which will be used throughout this thesis in the context of credit risk modelling.

2.3.1. Poisson Process. A Poisson process N_t is a unit-jump increasing, right continuous stochastic process with stationary and independent increments and initial condition $N_0 = 0$ [13].

It can be shown that this general definition implies several useful properties of the distribution and jump times, collected in the following lemma.

Lemma 2.10. *Let τ be the first jump time of the Poisson process N_t. Then there exists a positive constant λ such that*

a) $\lim_{dt\to 0} \frac{\mathbb{P}(\tau \in (t, t+dt) \mid \tau > t)}{dt} = \lambda$ *- that is, the instantaneous conditional jump probability in an interval of length dt is approximately $\lambda\, dt$.*

b) $\mathbb{P}(N_t = 0) = \mathbb{P}(\tau > t) = e^{-\lambda t}$ *for all t.*

c) $\lambda = \frac{\mathbb{E}(N_t)}{t} = \frac{Var(N_t)}{t}$*, the average arrival rate and variance per unit of time.*

d) The random variable τ is distributed according to an exponential distribution with parameter λ, that is,

$$\tau \sim \textsc{Exponential}(\lambda)$$

e) Finally, the distribution of the process N_t is Poisson with parameter λt, and hence

$$N_t - N_s \sim \textsc{Poisson}\left((t-s)\lambda\right).$$

The Poisson process is the simplest pure jump process, with Lévy triplet $(0, 0, \lambda\delta_1)$, where δ_1 is the Dirac measure at 1 and λ is the positive constant defined in the previous lemma..

Figure 2.1 shows a sample path of a Poisson process.

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[1] If $\nu\left(\mathbb{R}\backslash(-1,1)\right) = 0$, then simply set $Y_t^{(2)} = 0$ for all t.

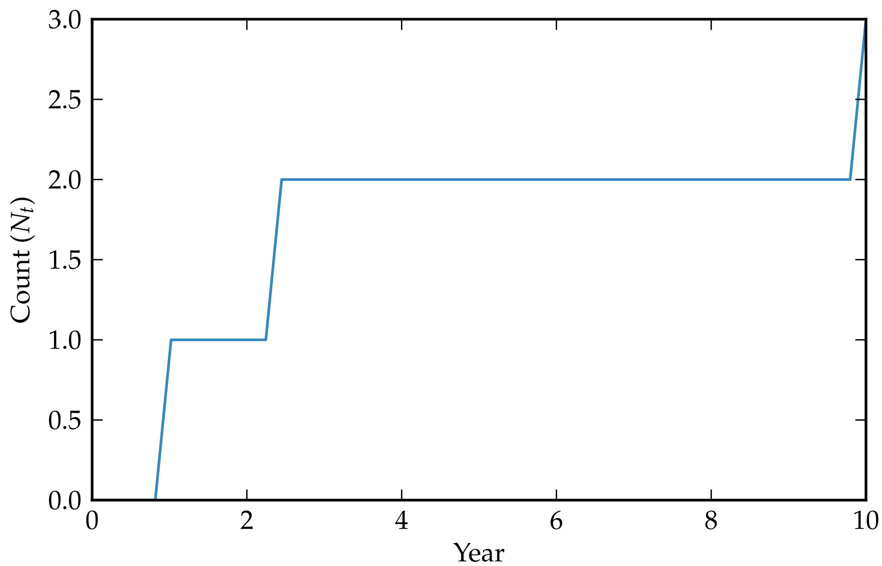

FIGURE 2.1. Poisson process with parameter $\lambda = 1.5$.

2.3.2. The Gamma Process. We first define the Gamma distribution, which is characterised by two positive parameters $a, b > 0$. The density function of the GAMMA(a, b) distribution is given by

$$f_{Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}, x > 0. \tag{2.11}$$

From (2.11), we can write the characteristic function of a GAMMA(a, b) distribution as

$$\varphi_{Gamma}(u; a, b) = \left(1 - \frac{iu}{b}\right)^{-a}$$

which is infinitely divisible. By Theorem 2.3, we can then form the *Gamma process*.

Definition 2.12 (Gamma process). *A stochastic process X_t is a Gamma process with parameters a and b if the following hold:*

a) $X_0 = 0$.
b) The process has independent, stationary, increments.
c) For any $0 < s < t$ *the increment* $X_t - X_s$ *has a* GAMMA$(a(t-s), b)$ *distribution.*

This is a Lévy process with a Lévy triplet given by

$$\left(\frac{a}{b}\left(1 - e^{-b}\right), 0, \frac{ae^{-bx}}{x} 1_{\{x>0\}}\, dx\right)$$

2.3.3. Inverse Gaussian Process. Consider a Brownian motion with drift $B_t = bt + W_t$ where $b > 0$ and W_t is a standard Brownian motion. Let

$$T_{a,b} = \inf_{t>0} \{B_t > a\}$$

for $a > 0$, that is, be the first time a Brownian motion with drift b reaches the positive level $a > 0$. Then the random variable $T_{a,b}$ follows an Inverse Gaussian law $\text{INVERSEGAUSSIAN}(a, b)$.

Following the same method as with the Gamma process, we can show that the Inverse Gaussian distribution is infinitely divisible, and thus we can define an *Inverse Gaussian process*.

Definition 2.13 (Inverse Gaussian process). *A stochastic process X_t is an Inverse Gaussian process with parameters $a, b > 0$ if the following hold:*

a) $X_0 = 0$.
b) The process has independent, stationary, increments.
c) For any $0 < s < t$, the increment $X_t - X_s$ has a $\text{INVERSEGAUSSIAN}(a(t-s), b)$ distribution.

2.4. Ornstein-Uhlenbeck Processes

Ornstein-Uhlenbeck processes were first developed by the physicists Orstein and Uhlenbeck in the early 20$^{\text{th}}$ century, in the context of integrating frictional effects with Brownian motion in statistical physics. In financial mathematics, Ornstein-Uhlenbeck processes were first used in modelling interest rates, in the seminal paper of [63], and more recently have been applied in modelling financial volatility [5, 6] and credit risk [16, 57].

Definition 2.14 (Ornstein-Uhlenbeck process [5]). *An Ornstein-Uhlenbeck (OU) process y_t is a stochastic process by the following stochastic differential equation:*

$$dy_t = -\lambda y_t\, dt + dz_{\lambda t}, \quad y_0 > 0, \tag{2.15}$$

where λ is an arbitrary positive parameter and z_t is a subordinator *- a Lévy process with no Brownian component, non-negative drift, and only positive jumps.*[2] *The subordinator z_t is known as the Background Driving Lévy Process (BDLP) [5].*

Equivalently, a stationary process y_t is an OU process if

$$y_t = e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)}\, dz_{\lambda s}. \tag{2.16}$$

Note that y_t is strictly positive, and is bounded below by the deterministic function $y_0 e^{-\lambda t}$.

A related process often studied in the context of credit modelling using OU processes is the integrated OU process (the intOU process) Y_t, defined by:

$$Y_t = \int_0^t y_s\, ds.$$

We note the following result, given in [16].

[2]It is possible to define OU processes where the BDLP is a general Lévy process, but we only consider the special case from above.

Theorem 2.17. *Let y_t be the OU process solving* (2.15)*. Let Y_t be the associated integrated OU process. Then*

$$\begin{aligned} Y_t &= \frac{1}{\lambda}\left(z_{\lambda t} - y_t + y_0\right) \\ &= \frac{1}{\lambda}\left(1 - e^{-\lambda t}\right) y_0 + \frac{1}{\lambda}\int_0^t \left(1 - e^{-\lambda(t-s)}\right)\, dz_{\lambda s} \end{aligned}$$

Proof. Let $T > 0$. Then from (2.16) we have

$$\begin{aligned} Y_T &= \int_0^T y_t\, dt \\ &= \int_0^T \left(e^{-\lambda t} y_0 + \int_0^t e^{-\lambda(t-s)}\, dz_{\lambda s}\right)\, dt \end{aligned}$$

Treating the first term, we have

$$\int_0^T e^{-\lambda t} y_0\, dt = \frac{1}{\lambda}\left(1 - e^{-\lambda T}\right) y_0.$$

Treating the second term, we have

$$\begin{aligned} \int_0^T \int_0^t e^{-\lambda(t-s)}\, dz_{\lambda s}\, dt &= \int_0^T \int_0^t e^{-\lambda t} e^{\lambda s}\, dz_{\lambda s}\, dt \\ &= \int_0^T \int_s^T e^{-\lambda t} e^{\lambda s}\, dt\, dz_{\lambda s} \quad \text{by the stochastic Fubini theorem} \\ &= \int_0^T e^{\lambda s} \int_s^T e^{-\lambda t}\, dt\, dz_{\lambda s} \\ &= \frac{1}{\lambda}\int_0^T \left(1 - e^{-\lambda(t-s)}\right) dz_{\lambda s} \end{aligned}$$

Combining these results, we have

$$Y_T = \frac{1}{\lambda}\left(1 - e^{-\lambda T}\right) y_0 + \frac{1}{\lambda}\int_0^T \left(1 - e^{-\lambda(t-s)}\right) dz_{\lambda s}$$

and our theorem is proven. □

Several major developments in the theory of OU processes driven by Lévy processes come from Sato [54], which we will now quote. We first need the concept of *self-decomposability*.

Definition 2.18 (Self-decomposability)**.** *Let φ be the characteristic function of a random variable X. Then X is self-decomposable if there exists a family of characteristic functions φ_c such that*

$$\varphi(t) = \varphi(ct)\varphi_c(t)$$

for all $t \in \mathbb{R}$ and all $c \in (0, 1)$.

Alternatively, a random variable X is said to be self-decomposable if for any $0 < c < 1$*, there exists a random variable* Y_c*, independent of X, such that*

$$X \sim cX + Y_c.$$

Example 2.19. The following distributions have all been shown to be self-decomposable:

(i) The $\textsc{Gamma}(a, b)$ distribution (see [35]),
(ii) The $\textsc{InverseGaussian}(a, b)$ distribution (see [35]),

Sato [54] establishes a fundamental relationship between *self-decomposable* random variables and the stationarity of OU processes, given below.

Theorem 2.20 (Theorem 1 of [5])**.** *Let* $\lambda > 0$ *and let D be a self-decomposable random variable. Then there exists a stationary stochastic process* y_t *and a Lévy process* z_t *such that* $y_t \sim D$ *and* y_t *satisfies the Ornstein-Uhlenbeck differential equation*

$$dy_t = -\lambda y_t\, dt + dz_{\lambda t}.$$

We call an OU process with stationary law D a D-OU process, drawing on notation from [5].

Thus, given a self-decomposable distribution D, we can construct an OU process with a stationary distribution D. We proceed to do this explicitly for the Gamma and InverseGaussian distributions.

2.4.1. Gamma-OU Processes. An important class of OU processes is the Gamma-OU process, where the stationary law is given by a $\textsc{Gamma}(a, b)$ distribution. It can be shown (see [61]) that the associated BDLP z_t is a compound Poisson process

$$z_t = \sum_{n=1}^{N_t} w_n$$

where N_t is a Poisson process with intensity a and w_n is a sequence of independent identically distributed $\textsc{Exponential}(b)$ random variables.

If y_t is a Gamma-OU process, the characteristic function of the intOU process $Y_t = \int_0^t y_s\, ds$ is given by:

$$\begin{aligned} \varphi_{G-OU}(u, t; \lambda, a, b, y_0) &= \mathbb{E}(\exp(iuY_t) \mid y_0) \\ &= \exp\left(\frac{iuy_0}{\lambda}(1 - e^{-\lambda t}) + \frac{\lambda a}{iu - \lambda b} \right. \\ &\qquad \left. \times \left(b \log\left(\frac{b}{b - iu\lambda^{-1}(1 - e^{-\lambda t})} \right) - iut \right) \right). \end{aligned} \tag{2.21}$$

A derivation is given in [51].

Figure 2.2 shows a sample path of the Gamma-OU process.

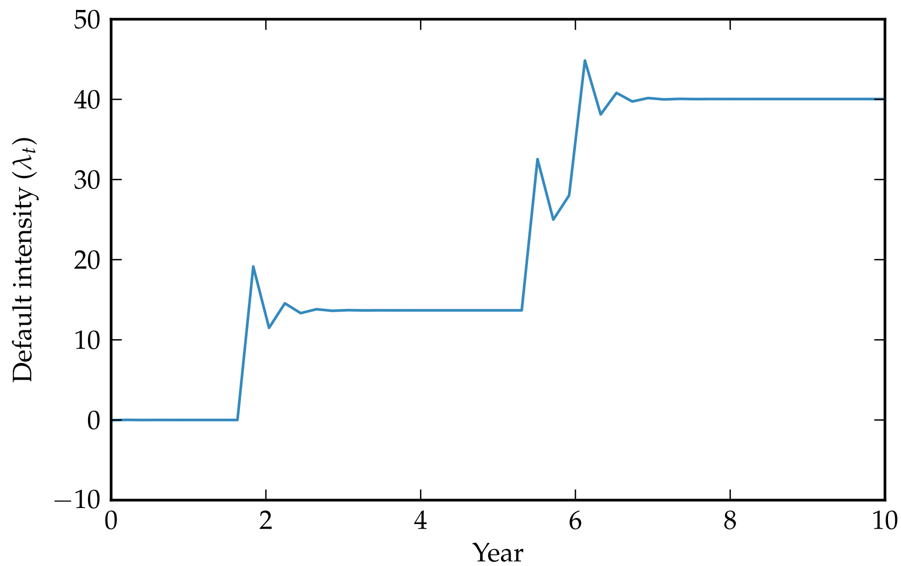

Figure 2.2. Gamma-OU process with parameters $\lambda = 2, a = 0.2, b = 18, y_0 = 0.08$.

2.4.2. Inverse Gaussian-OU Processes. We can similarly define the Inverse Gaussian-OU process (IG-OU), where the stationary law is given by a InverseGaussian(a, b) distribution. It can be shown that the BDLP is a sum of two independent Lévy processes $z_t^{(1)}$ and $z_t^{(2)}$. $z_t^{(1)}$ is an Inverse Gaussian process with parameters $a/2$ and b, and $z_t^{(2)}$ is of the form

$$z_t^{(2)} = \frac{1}{b} \sum_{n=1}^{N_t} v_n^2,$$

where N_t is a Poisson process with intensity $ab/2$ and v_n is a sequence of independent identically distributed Normal$(0, 1)$ random variables.

If y_t is an IG-OU process, the characteristic function of the intOU process $Y_t = \int_0^t y_s \, ds$ is given by

$$\begin{aligned} \varphi_{IG-OU}(u, t; \lambda, a, b, y_0) &= \mathbb{E}(\exp(iuY_t) \mid y_0) \\ &= \exp\left(\frac{iuy_0}{\lambda}(1 - e^{-\lambda t}) + \frac{2aiu}{b\lambda} A(u, t) \right) \end{aligned} \tag{2.22}$$

where

$$\begin{aligned} A(u, t) = \frac{1 - \sqrt{1 + \kappa(1 - e^{-\lambda t})}}{\kappa} + \frac{1}{\sqrt{1 + \kappa}} \\ \times \left[\operatorname{arctanh}\left(\frac{\sqrt{1 + \kappa(1 - e^{-\lambda t})}}{\sqrt{1 + \kappa}} \right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1 + \kappa}} \right) \right] \end{aligned} \tag{2.23}$$

A derivation is given in [51].

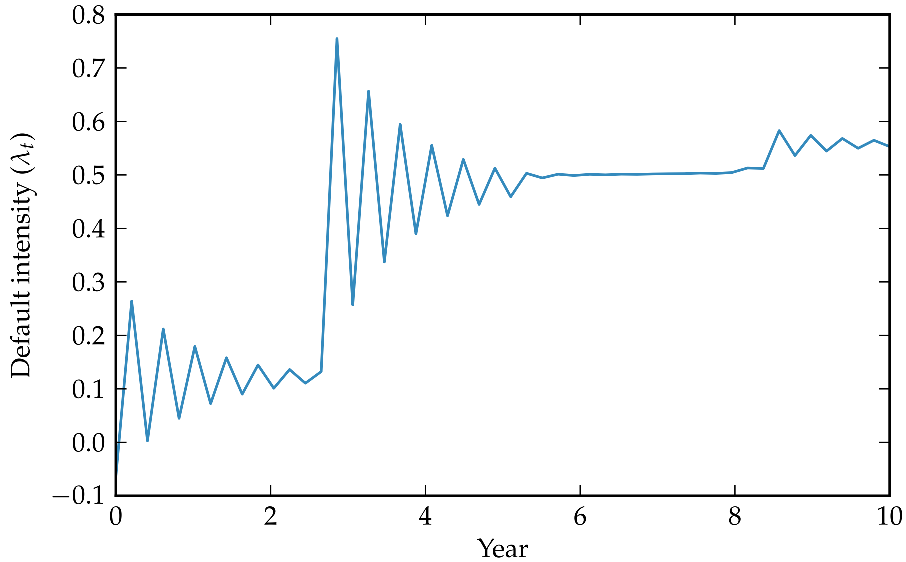

Figure 2.3. Inverse Gaussian-OU process with parameters $\lambda = 4, a = 0.2, b = 5, y_0 = 0.08$.

Figure 2.3 shows a sample path of the Gamma-OU process.

Chapter 3

Pricing Credit Default Swaps

3.1. Survival Probability Modelling

As discussed in §1.1, the payoff from a credit default swap is dependent on the survival or default of the reference entity. Thus, accurately modelling the probability of a reference entity defaulting in a time interval is of key interest to market participants. We now focus on modelling this probability in an intensity framework.

3.1.1. Canonical Construction of Default Times. We now introduce the most common construction of a default time in an intensity framework.[1] Assume that we are given a $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Assume that $\Gamma(0) = 0$ and $\Gamma(\infty) = +\infty$. In our framework, Γ will be given by

$$\Gamma_t = \int_0^t \lambda_u \, du, \quad t > 0$$

for some non-negative, $\mathbb{F}$-progressively measurable intensity process λ_t.

Let χ be a uniform random variable on $[0,1]$, independent of the filtration $\mathbb{F}$ under $\mathbb{P}$. We can then define the random time $\tau : \Omega \to \mathbb{R}^+$ by the formula

$$\tau = \inf\{t \in R^+ \mid e^{-\Gamma_t} \leq \chi\} = \inf\{t \in \mathbb{R}^+ \mid \Gamma_t \geq \xi\} \tag{3.1}$$

where $\xi = -\ln \chi$ is a standard exponential random variable.

We then obtain expressions for *survival probabilities*, defined as

$$\mathbb{P}_{Surv}(t) = \mathbb{P}(\tau > t) \tag{3.2}$$

as follows. Since $\{\tau > t\} = \{\chi < e^{-\Gamma_t}\}$ and Γ_t is $\mathcal{F}_\infty$-measurable, we have

$$\mathbb{P}(\tau > t \mid \mathcal{F}_\infty) = \mathbb{P}(\chi < e^{-\Gamma_t} \mid \mathcal{F}_\infty) = e^{-\Gamma_t}$$

and hence

$$\mathbb{P}(\tau > t \mid \mathcal{F}_t) = \mathbb{E}(\mathbb{P}(\tau > t \mid \mathcal{F}_\infty) \mid \mathcal{F}_t) = e^{-\Gamma_t}$$

Taking expectations with respect to $\mathbb{P}$, we obtain our key result,

$$\mathbb{P}_{Surv}(t) = \mathbb{P}(\tau > t) = \mathbb{E}\left(\mathbb{P}(\tau > t \mid \mathcal{F}_t)\right) = \mathbb{E}\left(e^{-\Gamma_t}\right) \tag{3.3}$$

[1] This *canonical construction* is not the most general method for obtaining random times associated with a given hazard process Γ, but is sufficient for our purposes.

The rest of this section follows by introducing various specifications for our intensity process λ_t, and deriving expressions for the survival probabilities in these models.

3.1.2. Time Homogenous Poisson Processes. In the simplest possible intensity model, we assume that the intensity process λ_t is equal to a constant λ. Our hazard process becomes

$$\Gamma_t = \int_0^t \lambda \, dt = \lambda t.$$

We obtain an explicit formula for $\mathbb{P}_{Surv}(t)$, given as

$$\mathbb{P}_{Surv}(t) = e^{-\Gamma_t} = e^{-\lambda t}.$$

3.1.3. Inhomgenous Poisson Processes. Now assume that the intensity parameter λ_t is a strictly positive, piecewise continuous, and deterministic. From (3.3), we obtain

$$\mathbb{P}_{Surv}(t) = e^{-\Gamma_t} = e^{-\int_0^t \lambda_t \, dt}.$$

3.1.4. Cox Processes. The previous models assumed that the default intensity λ_t was a deterministic function of time. Allowing λ_t to follow a stochastic process leads us to the study of Cox processes. The study of the use of Cox processes in modelling default risk was instigated by [45], with extensions and generalisations discussed in [25, 26].

Definition 3.4 (Cox [20]). *Let λ_t be a strictly positive, $\mathcal{F}_t$-adapted, right continuous function. Then a Poisson process with stochastic intensity λ_t is known as a Cox process, also referred to as a doubly stochastic Poisson process.*

For Cox processes with parameter λ_t, the conditions on λ_t imply that the random variable

$$\Gamma_t = \int_0^t \lambda_u \, du.$$

is well defined. We notes that conditioning on the filtration generated by λ_t, denoted $\mathcal{F}_t^\lambda$, a Cox process becomes an inhomogenous Poisson process with intensity λ_t.

We can then derive expressions for survival probabilities in the Cox model. We have

$$\mathbb{P}_{Surv}(t) = \mathbb{P}(\tau > t) = \mathbb{E}\left(e^{-\Gamma_t}\right)$$

3.2. Credit Default Swaps

The credit default swap is the most commonly traded credit derivative. As discussed in §1.1, a credit default swap allows two parties, the *protection seller* and *protection buyer* to exchange the default risk on a specified reference entity. Typically, the protection seller receives quarterly payments (the *CDS spread*), in return for protecting the protection buyer against the default of the reference entity. The cash-flow diagram is presented in Figure 3.1, and a typical sequence of payments in a CDS is presented in Figure 3.2.

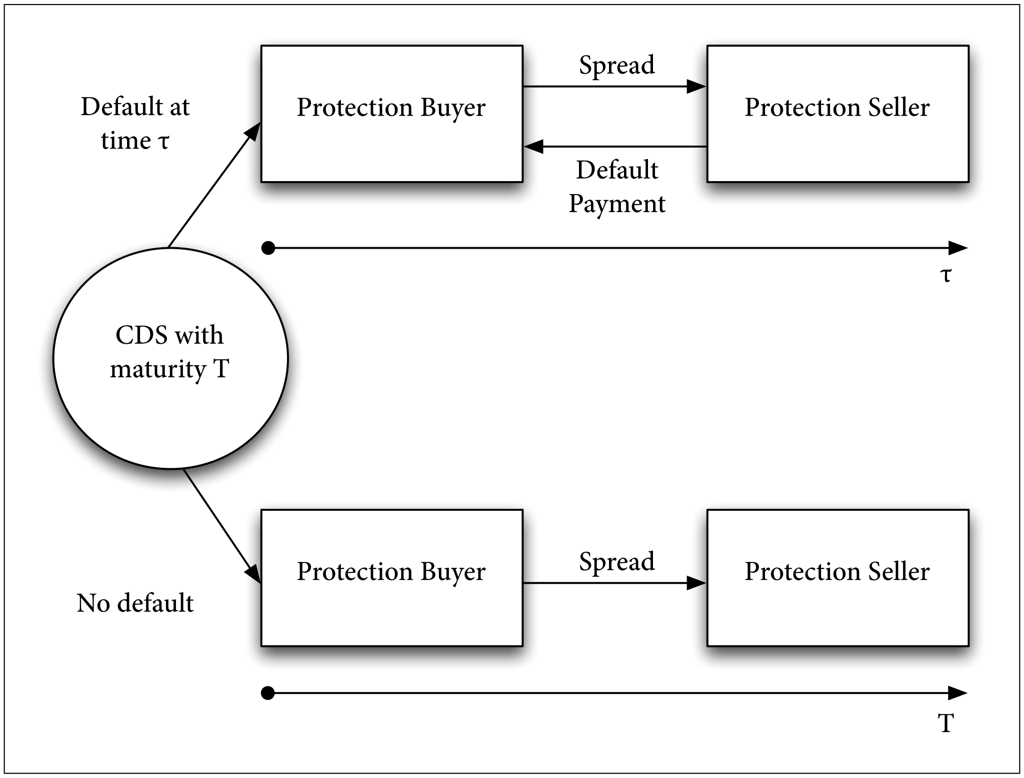

FIGURE 3.1. Schematic diagram of a CDS. In the event of a default, the protection seller pays the protection buyer, and the protection buyer periodically pays the CDS spread to the protection seller.

More specifically, consider a CDS with a notional of N, with a tenor of two years ($T = 2$). Let the CDS spread be denoted by c. Then at the end of every quarter, the protection buyer would pay $\frac{1}{4}cN$ to the seller. There are then two cases:

- If the reference entity does not default before maturity, the protection buyer simply continues to pay the quarterly payments to the seller, and the swap expires after two years.
- If the reference entity does default before maturity, the protection seller must pay the buyer an amount to compensate for the default. The default payment is equal to the difference between the notional value and the recovered amount $N \cdot R$ after default.[2] That is, the protection seller will pay $N(1 - R)$ to the buyer, after which the swap terminates and no further payments are made.

3.3. CDS Pricing in Discrete Time

We now turn our attention to pricing a CDS with maturity T - that is, finding the value of the spread c such that the swap costs nothing at time $t = 0$ to enter into. For simplicity, let us consider a

[2]When an entity defaults, investors typically receive some proportion of their investment back as the entity's assets are liquidated. Typical recovery rates R are around 50% for senior secured debt-holders and 20% for junior subordinated debt-holders.

discrete setting, where the payments and default events occur only at discrete times t_i, $i = 1, 2, \dots, n$ and $t_n = T$. For convenience, set $t_0 = 0$. Figure 3.3 illustrates the discrete nature of our pricing model.

Let $P(0, t_i)$ be the discount factor from time 0 to time t_i, defined in (1.1). The short rate r_t can be stochastic or deterministic, but we require r_t to be uncorrelated with the intensity λ_t. Let Δt_i be the time difference between two consecutive payouts ($\Delta t_i = t_i - t_{i-1}$). Let the default time of the reference entity be τ.

Then the present value of the cash-flows of the fee leg are equal to

$$CF_{Fees} = cN \sum_{i=1}^{n} P(0, t_i) \Delta t_i 1_{\tau > t_i} \tag{3.5}$$

and the present value of the cash-flows of the loss leg are equal to

$$CF_{Loss} = (1 - R)N \sum_{i=1}^{n} P(0, t_i)(1_{\tau = t_i}). \tag{3.6}$$

Taking the risk-neutral expectation of the cash-flows of both legs gives

$$PV_{Fees} = cN \sum_{i=1}^{n} P(0, t_i) \mathbb{P}_{Surv}(t_i) \Delta t_i$$

$$PV_{Loss} = (1 - R)N \sum_{i=1}^{n} P(0, t_i) \left(\mathbb{P}_{Surv}(t_{i-1}) - \mathbb{P}_{Surv}(t_i) \right).$$

where $\mathbb{P}_{Surv}(t_i)$ indicates the survival probability up to time t_i (defined in (3.3)) and c is the CDS spread per annum.

Thus, in our discrete setting, the par spread c which makes the value of the fee leg equal to the value of the loss leg is given by

$$c = \frac{(1 - R) \sum_{i=1}^{n} P(0, t_i) \left(\mathbb{P}_{Surv}(t_{i-1}) - \mathbb{P}_{Surv}(t_i) \right)}{\sum_{i=1}^{n} P(0, t_i) \mathbb{P}_{Surv}(t_i) \Delta t_i}. \tag{3.7}$$

3.4. CDS Pricing in Continuous Time

We now move to the problem of pricing CDS contracts in continuous time, following [13]. When moving to continuous time pricing, we need to consider the possibility of default occurring at some time in (t_{i-1}, t_i). Most commonly, CDS contracts specify that the protection payment occurs immediately following default. This standard form of CDS is known as a *running CDS*.

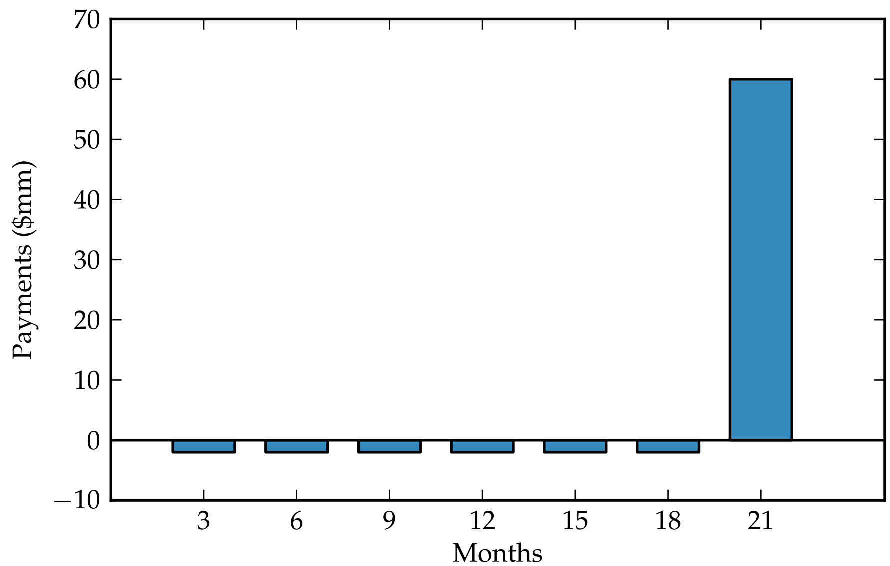

FIGURE 3.2. Cash-flows of a CDS from the perspective of the protection buyer. Here, we assume a notional of $100mm, a tenor of 2 years, and a CDS spread of 200 bps (2.00%). We assume default occurs after 21 months, and the recovery rate is 40%.

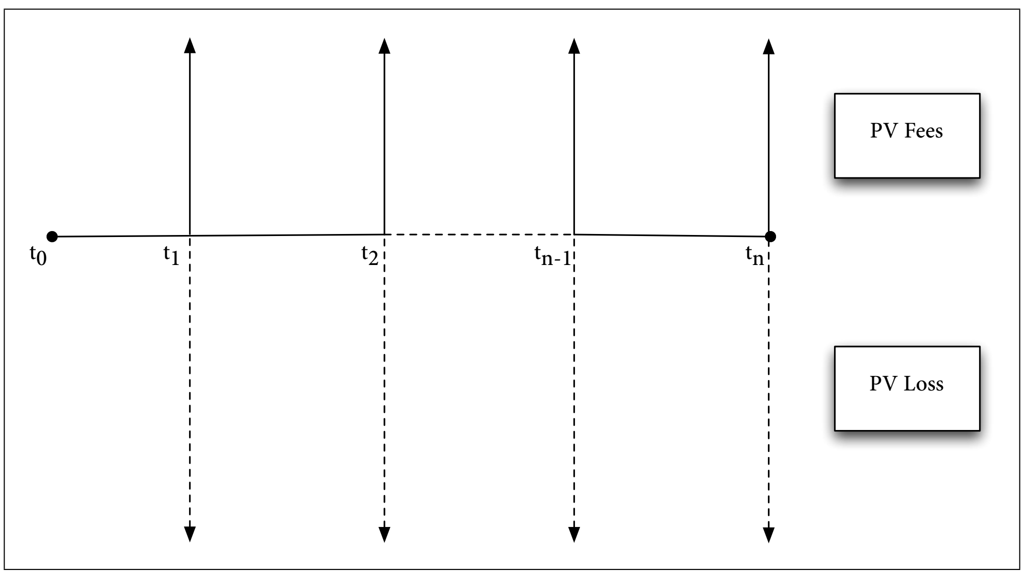

FIGURE 3.3. Timeline of the discrete CDS model. At each discrete time $t_i, i = 1, \ldots, n$, the reference entity either defaults (in which case the protection seller pays the buyer), or does not default (in which case the protection buyer pays the seller)

Under the same assumptions and following the same process as in the discrete case, we can write the discounted value of cash-flows of the fee and protection legs at time $t < T$ as

$$\Pi_{Fees}(t, c) = cNP(t, \tau)(\tau - t_{\beta(\tau)-1})1_{\{0<\tau<T\}} + cN\sum_{i=1}^{n} P(t, t_i)\Delta t_i 1_{\{\tau \geq t_i\}}$$

$$\Pi_{Loss}(t, c) = (1 - R)NP(t, \tau)1_{\{0<\tau\leq T\}}$$

where τ is the default time, c is the CDS spread per annum, N is the contract notional, $\Delta t_i = t_i - t_{i-1}$ is the year fraction for the payment period, and $t_{\beta(t)}$ is the first date in t_i, $i = 1, 2, \ldots$ such that $t_i > t$. The first term in the Π_{Fees} expression is the *accrued interest*, reflecting the possibility of default in (t_{i-1}, t_i).

From elementary martingale pricing theory, we know that the price of a contingent claim is equal to the discounted risk-neutral conditional expectation of the payoff (see [7, 13, 49] for further detail). Thus, the value of a CDS contract with tenor T and spread c, at time $t < T$, denoted $\mathbf{CDS}_t$, as seen by the protection seller, is

$$\mathbf{CDS}_t = \mathbb{E}\left(\Pi_{Fees}(t, c) - \Pi_{Loss}(t, c) \,|\, \mathcal{G}_t\right). \tag{3.8}$$

Here, $\mathcal{G}_t$ is our default-free filtration $\mathcal{F}_t$ augmented by a default monitoring process, as discussed in §1.3.

In some cases it may be advantageous to compute this expectation with respect to the default-free filtration $\mathcal{F}_t$. We now introduce and prove this *filtration switching* formula.

Proposition 3.9 (Filtration switching formula)**.** *Let $\mathcal{F}_t$ and $\mathcal{G}_t$ be defined as previously. Let X be a $\mathcal{G}_\infty$ measurable payoff. Let $t < T$. Then we have*

$$\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{G}_t) = \frac{1_{\{\tau>t\}}}{\mathbb{P}\left(\tau > t \mid \mathcal{F}_t\right)}\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{F}_t) \tag{3.10}$$

Proof. Here, we follow [13]. For a proof under more general conditions, see [8].

First, we recall that

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \le u\}, u \le t)$$

We can write

$$\begin{aligned}
\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{G}_t) &= \mathbb{E}(1_{\{\tau>t\}}1_{\{\tau>T\}}X \,|\, \mathcal{G}_t) \\
&= 1_{\{\tau>t\}}\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{G}_t) \\
&= 1_{\{\tau>t\}}\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{F}_t, \{\tau > t\})
\end{aligned}$$

as the only useful information in $\sigma(\{\tau \le u\}, u \le t$ is whether $\tau > t$. This is because $1_{\{\tau>T\}}X$ contributes nothing to the conditional expectation when $\tau \le T$.

By the definition of conditional expectation, we then have

$$\begin{aligned}
1_{\{\tau>t\}}\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{F}_t, \{\tau > t\}) &= \frac{1_{\{\tau>t\}}}{\mathbb{P}\left(\tau > t \mid \mathcal{F}_t\right)}\mathbb{E}(1_{\{\tau>t\}}1_{\{\tau>T\}}X \,|\, \mathcal{F}_t) \\
&= \frac{1_{\{\tau>t\}}}{\mathbb{P}\left(\tau > t \mid \mathcal{F}_t\right)}\mathbb{E}(1_{\{\tau>T\}}X \,|\, \mathcal{F}_t)
\end{aligned}$$

where $\mathbb{P}(\tau > t \mid \mathcal{F}_t)$ is the risk-neutral probability that the entity will survive up to time t, conditional on default-free information up to time t. Thus

$$\mathbb{E}(1_{\{\tau>t\}}X \mid \mathcal{G}_t) = \frac{1_{\{\tau>T\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\mathbb{E}(1_{\{\tau>T\}}X \mid \mathcal{F}_t)$$

as required. □

Using this formula we just derived, we can write the expectation in (3.8) as

$$\mathbf{CDS}_t = \frac{1_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathcal{F}_t)}\mathbb{E}\left(\Pi_{Fees}(t,c) - \Pi_{Loss}(t,c) \mid \mathcal{F}_t\right)$$

which allows us to calculate the par spread of a running CDS with an expectation using the default-free filtration.

3.5. Term Structure of CDS Contracts

Consider a *continuous CDS*, in which a spread c is paid continuously for the tenor T of the contract, with payments ceasing at either an event of default τ or at maturity T. This is the simplest continuous time generalisation of the discrete time model in §3.3. We easily see that the continuous yearly par spread c_{cts} is a straightforward generalisation of (3.7) given by

$$c_{cts} = \frac{(1-R)\left(-\int_0^T P(0,s)\, d\mathbb{P}_{Surv}(s)\right)}{\int_0^T P(0,s)\mathbb{P}_{Surv}(s)\, ds} \tag{3.11}$$

This analytically tractable model allows us to derive an approximate relationship between the intensity rate and the CDS spread. Consider the model discussed previously in §3.1.2 - the homogenous Poisson model first introduced in [39]. Here, we assume the default time is exponentially distributed with parameter λ, and so the expected default time $\mathbb{E}(\tau) = \frac{1}{\lambda}$. Then we have the survival probability $\mathbb{P}_{Surv}(t) = e^{-\lambda t}$. By substituting these results into (3.11), we can derive that the continuous yearly par spread c_{cts} is given by

$$\begin{aligned} c_{cts} &= \frac{(1-R)\left(-\int_0^T P(0,s)\lambda e^{-\lambda s}\, ds\right)}{\int_0^T P(0,s)e^{-\lambda s}\, ds} \\ &= (1-R)\lambda \end{aligned}$$

Thus, we see two important features of the homogenous Poisson model.

(i) Continuous CDS spreads are linear functions of the default intensity.
(ii) The homogenous Poisson model suggests the *term structure* - that is, the relationship between CDS spreads of different maturities - should be flat.

We shall return to these features in §4.2.1, and for now illustrate the relationship between CDS spreads of different maturities

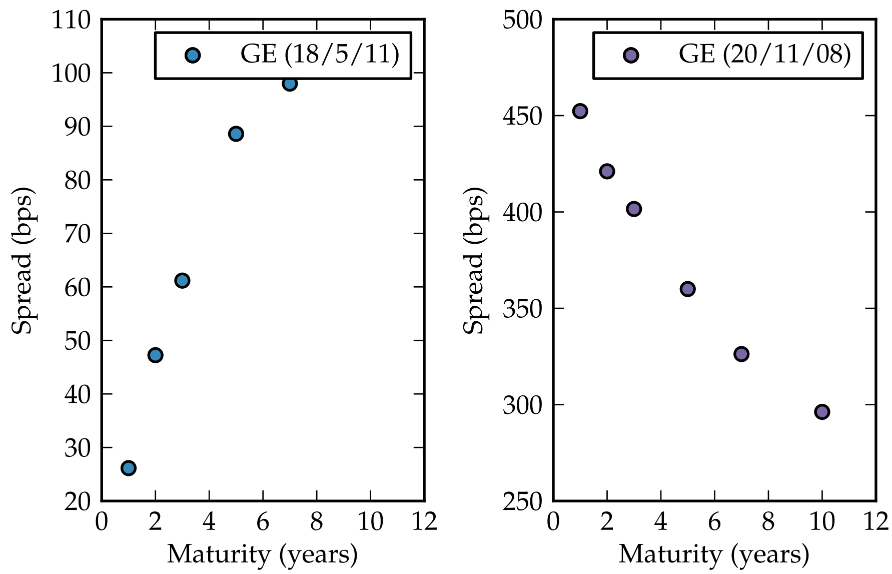

Figure 3.4. CDS term structure of General Electric after (left hand side), and during the height of the credit crunch (right hand side). Note the inversion of the term structure in times of crisis. *Source:* Goldman Sachs.

In the credit derivatives market, the most liquid CDS maturity is at 5 years, with trading occurring at {1y, 2y, 3y, 5y, 7y, 10y}. An example CDS term structure is displayed in Figure 3.4. Notice the upward sloping curve, which is commonly seen. When examining severely distressed names, we often observe a downward sloping curve, reflecting a higher probability of default in the short term.

Chapter 4

Empirical Investigation of Intensity Models

We now turn to an empirical examination of various intensity models for credit risk. Our goal is to calibrate a sequence of intensity models to the CDX index, and examine the calibration results. This is done in a single-name CDS context and for a limited subset of intensity models in [16].

4.1. Data and Methodology

Here we present results from calibrating various Lévy process as stochastic intensities to market data. The data set used for calibration is the term structure of General Electric CDS spreads from 5/10/07 to 6/10/11. General Electric is a multinational conglomerate, and is the most heavily traded corporation in the CDS market [37]. The dataset consists of 6 tenors (1y, 2y, 3y, 5y, 7y, 10y) and corresponding credit spreads. The data was obtained with permission from Goldman Sachs.

The interest rate r is assumed constant, and calibrated to the US Treasury curve at the close of 18/5/11. US Treasury data is obtained from Bloomberg. We test the following intensity models:

(1) The homogenous Poisson (HP) model, explained in §4.2.1.
(2) The inhomogenous Poisson (IHP) model, explained in §4.2.1.
(3) The Cox-Ingersoll-Ross model (CIR), explained in §4.2.2.
(4) The Gamma-OU model (G-OU), explained in §4.2.3.
(5) The Inverse Gaussian-OU model (IG-OU), explained in §4.2.3.

For computational reasons, we revert back to our discrete model of §3.3, and calculate the par spread as:

$$c = \frac{(1-R)\sum_{i=1}^{n} P(0,t_i)\left(\mathbb{P}_{Surv}(t_{i-1}) - \mathbb{P}_{Surv}(t_i)\right)}{\sum_{i=1}^{n} P(0,t_i)\mathbb{P}_{Surv}(t_i)\Delta t_i}. \tag{4.1}$$

All models are calibrated using the Nelder-Mead simplex algorithm, which is a robust method for optimising nonlinear functions [50]. Here, we seek to minimise the RMSE of our models, given by

$$\mathrm{RMSE}_{Model} = \sqrt{\sum^{N} \frac{|c_{Market} - c_{Model}|^2}{N}}$$

where N is the number of maturities observed in the market and c_{Market} and c_{Model} are the observed and calculated par spreads for the CDS respectively.

4.2. Intensity Models

4.2.1. Homogenous and Inhomogenous Poisson Models. The simplest possible intensity model is one where the intensity process λ_t is equal to a constant $\lambda > 0$, with the default time now following a *homogenous Poisson process*. This model, first explored in [39], was previously discussed in §3.1.2. There, we easily derived our expression for the survival probability

$$\mathbb{P}^{HP}_{Surv}(t) = e^{-\lambda t}.$$

Given this expression for the survival probability at any time t, we can then extract the implied par CDS spread from (3.7),

$$c = \frac{(1-R)\sum_{i=1}^{n} P(0,t_i)\left(\mathbb{P}_{Surv}(t_{i-1}) - \mathbb{P}_{Surv}(t_i)\right)}{\sum_{i=1}^{n} P(0,t_i)\mathbb{P}_{Surv}(t_i)\Delta t_i}.$$

As we saw in §3.5, this model predicts a flat term structure for CDS spreads, which is clearly refuted by market observations. Acknowledging this deficit, we now turn to the *inhomogenous Poisson process*, defined in §3.1.3. Here, the intensity process λ_t is assumed to be a deterministic function of time.

As before, the probability of survival past time t in the inhomogenous model now given by

$$\mathbb{P}^{IHP}_{Surv}(t) = \exp\left(-\int_0^t \lambda_s \, ds\right).$$

A fundamental example is where we take the intensity λ_t to be deterministic and piecewise constant:

$$\lambda_t = \gamma_i \quad \text{for } t \in [T_{i-1}, T_i], 1 \le i \le n$$

where the T_i span the maturities we seek to calibrate our model to.

This model is often used to extract risk-neutral market implied default probabilities [13]. In the piecewise constant case, we have as many free parameters as data points, and so we can find γ_i, $1 \le i \le n$, such that our calibrated model exactly reproduces the observed CDS prices in the market. We can then extract implied survival probabilities to each maturity T_i by our expression for $\mathbb{P}^{IHP}_{Surv}$ given above. An example of this process is given in Table 4.1.

4.2.2. Cox-Ingersoll-Ross Models. We now move to a stochastic framework, where we allow the process λ_t to follow a stochastic process. This corresponds to the Cox process, introduced in §3.1.4. There we showed that the survival probability $\mathbb{P}_{Surv}(t)$ satisfies

$$\mathbb{P}_{Surv}(t) = \mathbb{E}\left(e^{-\int_0^t \lambda_s \, ds}\right).$$

The paper of [21] introduced the Cox-Ingersoll-Ross (CIR) process λ_t, where λ_t satisfies the stochastic differential equation

$$d\lambda_t = \kappa(\nu - \lambda_t)\, dt + \gamma\sqrt{\lambda_t}\, dW_t, \quad \lambda_0 > 0. \tag{4.2}$$

We also apply the restriction $2\kappa\nu > \gamma^2$, which ensures the process is bounded away from zero.

Maturity (y)	Market (bps)	Model (bps)	Intensity	Survival Probability (%)
1	26	26	0.0044	99.6
2	47	47	0.0079	98.8
3	61	61	0.0102	97.8
5	89	89	0.0147	95.1
7	98	98	0.0163	92.9
10	105	105	0.0175	87.3

TABLE 4.1. The calibration of a piecewise-constant inhomogenous Poisson model to General Electric CDS spreads on 18/5/11. Notice how the model exactly reproduces the market spreads, and the survival probabilities are decreasing with time, as expected. *Source:* Goldman Sachs.

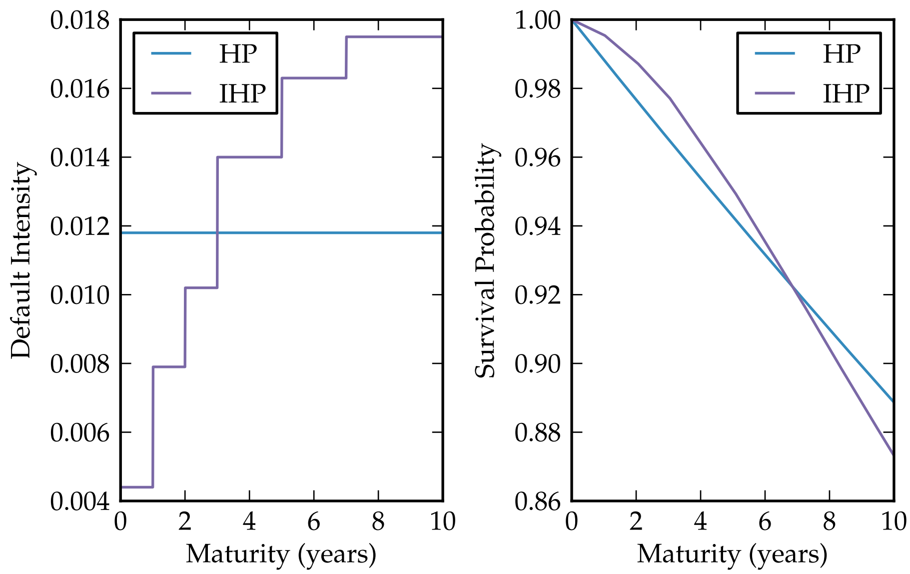

FIGURE 4.1. Default intensities (LHS) and survival probabilities (RHS) for calibrated homogenous Poisson (HP) and inhomogenous Poisson (IHP) models. Both models are calibrated to General Electric CDS spreads on 18/5/11. *Source:* Goldman Sachs.

If we assume the intensity rate follows the CIR dynamics above, then we can derive an expression for the survival probability $\mathbb{P}^{CIR}_{Surv}(t)$. By noting that

$$\mathbb{P}^{CIR}_{Surv}(t) = \mathbb{E}\left(\exp\left(-\int_0^t \lambda_s \, ds\right)\right) = \varphi_{CIR}(i, t)$$

where φ_{CIR} is the characteristic function of the integrated CIR process $\int_0^t \lambda_s \, ds$. Using the analytic expression for $\varphi_{CIR}(u,t)$ from [21], we obtain

$$\begin{aligned}\mathbb{P}^{CIR}_{Surv}(t) &= \varphi_{CIR}(i, t; \kappa, \nu, \gamma, \lambda_0) \\ &= \frac{\exp(\kappa^2 \nu t/\gamma^2)\exp(-2\gamma_0/(\kappa + \rho\coth(\rho t/2)))}{(\coth(\rho t/2) + \kappa\sinh(\rho t/2)/\rho)^{2\kappa\nu/\gamma^2}}\end{aligned}$$

where $\rho = \sqrt{\kappa^2 + 2\lambda_0^2}$.

4.2.3. Orstein-Uhlenbeck Models. We also consider Cox models where the intensity rate λ_t follows an Orstein-Uhlenbeck process, specifically the Gamma-OU (G-OU) and Inverse Gaussian-OU (IG-OU) models discussed in §4.2.3. Similar to the analysis above with the Cox process, we use the fact that

$$\mathbb{P}^{OU}_{Surv}(t) = \mathbb{E}\left(\exp\left(-\int_0^t \lambda_s \, ds\right)\right) = \varphi_{OU}(i,t)$$

where $\varphi_{OU}(u,t)$ is the characteristic function of the integrated OU process.

In the Gamma-OU case, from (2.21) and [58], we have:

$$\begin{aligned}\mathbb{P}^{G-OU}_{Surv}(t) &= \varphi_{G-OU}(i, t; \gamma, a, b, \lambda_0) \\ &= \exp\left(\frac{-\lambda_0}{\gamma}(1 - e^{-\gamma t}) - \frac{\gamma a}{1 + \gamma b}\right. \\ &\qquad \left. \times \left(b \log\left(\frac{b}{b + \frac{1}{\gamma}(1 - e^{-\gamma t})}\right) + t\right)\right).\end{aligned}$$

In the Inverse Gaussian-OU case, from (2.22) and [58], we have:

$$\begin{aligned}\mathbb{P}^{G-OU}_{Surv}(t) &= \varphi_{IG-OU}(i, t; \gamma, a, b, \lambda_0) \\ &= \exp\left(\frac{-\lambda_0}{\gamma}(1 - e^{-\gamma t}) - \frac{2a}{b}A(i,t)\right)\end{aligned}$$

where $A(u,t)$ is defined in (2.23).

4.3. Calibration Results

Our results for the 17/3/11 are detailed here. Figure 4.2 illustrates the calibrated term structures of the various models. Figure 4.3 illustrates the calibrated survival probabilities of the various models. Table 4.2 shows the calibration results and RMSE of the various models.

We note that the CIR, G-OU, and IG-OU models can be calibrated successfully to the CDS term structure, closely replicating the observed market data. The HP is clearly unsuitable for modelling the term structure of CDS contracts. The IHP model can be perfectly calibrated to market data, but suffers from poor parameter stability and an unrealistic term structure. Our conclusions broadly support those of [58].

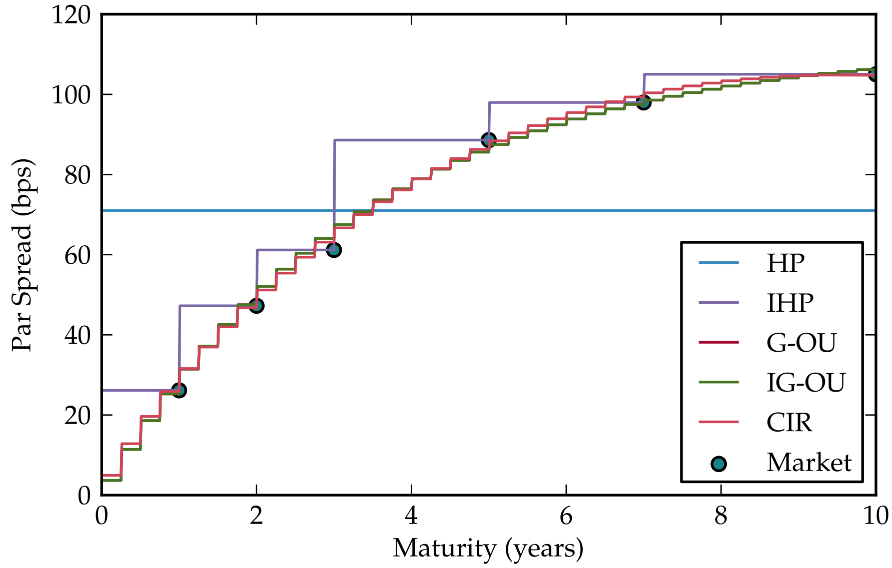

Figure 4.2. Term structure of the intensity models. All models are calibrated to General Electric CDS spreads on 18/5/11. *Source:* Goldman Sachs.

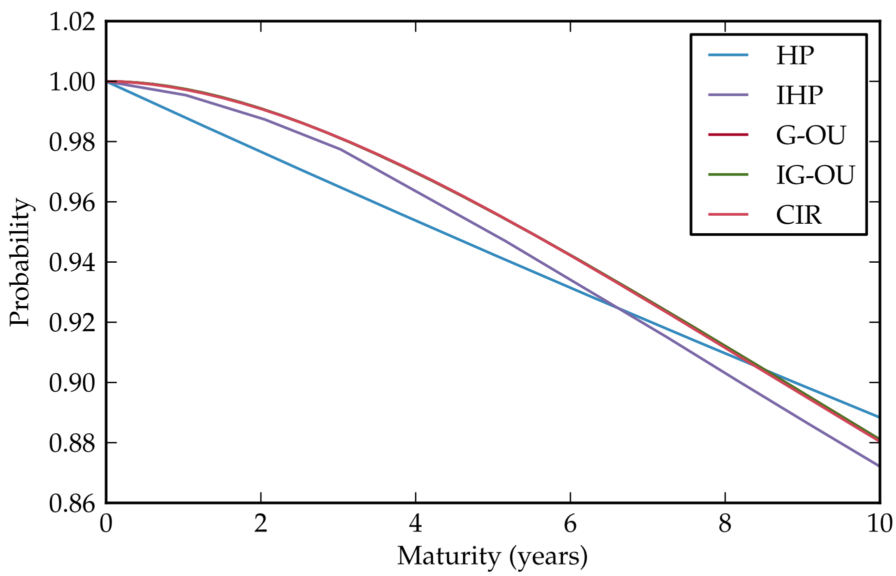

Figure 4.3. Survival probabilities of the intensity models. All models are calibrated to General Electric CDS spreads on 18/5/11. *Source:* Goldman Sachs.

	1y	2y	3y	5y	7y	10y	RMSE
Market	26	47	61	89	98	105	
HP	71	71	71	71	71	71	28.48
IHP	26	47	61	89	98	105	0.00
CIR	26	47	63	86	99	105	1.38
G-OU	26	48	64	86	98	106	1.82
IG-OU	25	48	64	86	97	106	1.82

Table 4.2. A comparison of the intensity models calibration to General Electric CDS spreads on 18/5/11. *Source:* Goldman Sachs.

Model	Parameters
HP	$\lambda = 0.0124$
IHP	$\lambda_0 = 0.0044, \lambda_1 = 0.0079, \lambda_2 = 0.0102, \lambda_3 = 0.0147, \lambda_4 = 0.0163, \lambda_5 = 0.0175$
CIR	$\kappa = 0.0180, \nu = 0.0175, \gamma = 0.0184, \lambda_0 = 0.0581$
G-OU	$\gamma = 0.2960, a = 70457, b = 3774752, \lambda_0 = 0.0001$
IG-OU	$\gamma = 0.2957, a = 31.98, b = 1713, \lambda_0 = 0.0001$

Table 4.3. Optimal calibration parameters for the intensity models. All models are calibrated to General Electric CDS spreads on 18/5/11. *Source:* Goldman Sachs.

4.4. Parameter Stability and RMSE

We now examine the stability of the calibrated parameters and the distribution of the RMSE through time for the CIR, Gamma-OU, and Inverse Gaussian-OU intensity models, following [15]. For each model, we consider two different methods for calibrating a series of CDS term structures, the *dynamic* method and the *static* method.

In the dynamic approach, the previous calibrated parameters are used as an initial guess for the calibration of the current day. This method often provides faster calibration times and greater parameter stability, yet often results in higher calibration errors.

In the static approach, the initial values for the calibrations are kept constant throughout the whole procedure. This avoids the optimisation being stuck in local minima, at a cost of longer calibration times.

Due to the pathological behaviour of the General Electric CDS term structure over the time period under consideration,[1] we consider the calibration of the intensity models to the CDX index.

The following results indicate that the Gamma-OU and Inverse Gaussian-OU intensity models can provide excellent fits to market data, with low RMSEs and high first-order parameter autocorrelations.

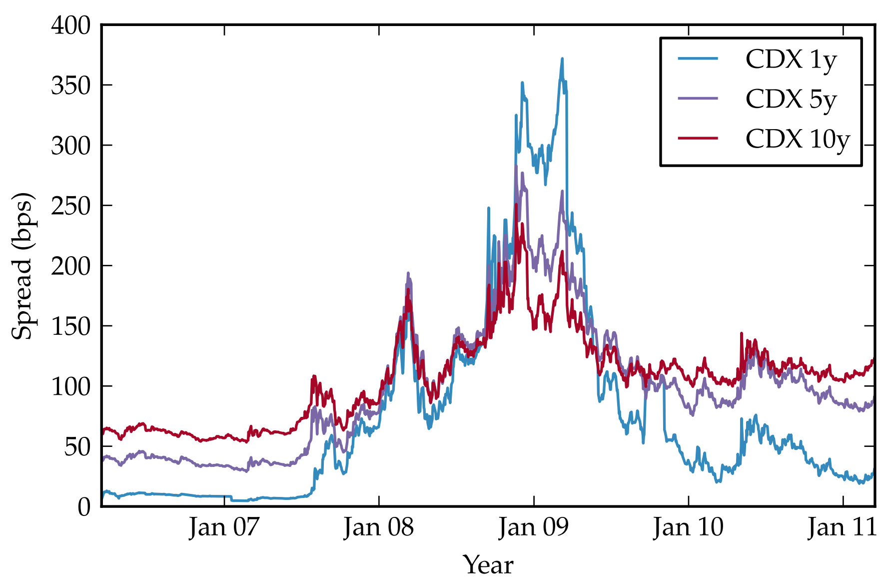

Figure 4.4. Performance of the CDX credit index from 5/10/2007 to 6/10/2011. Note that the curve inversion is much less pronounced as compared to GE CDS spreads. *Source:* Goldman Sachs.

[1]See Figure 1.2 for the historical term structure of GE CDS spreads.

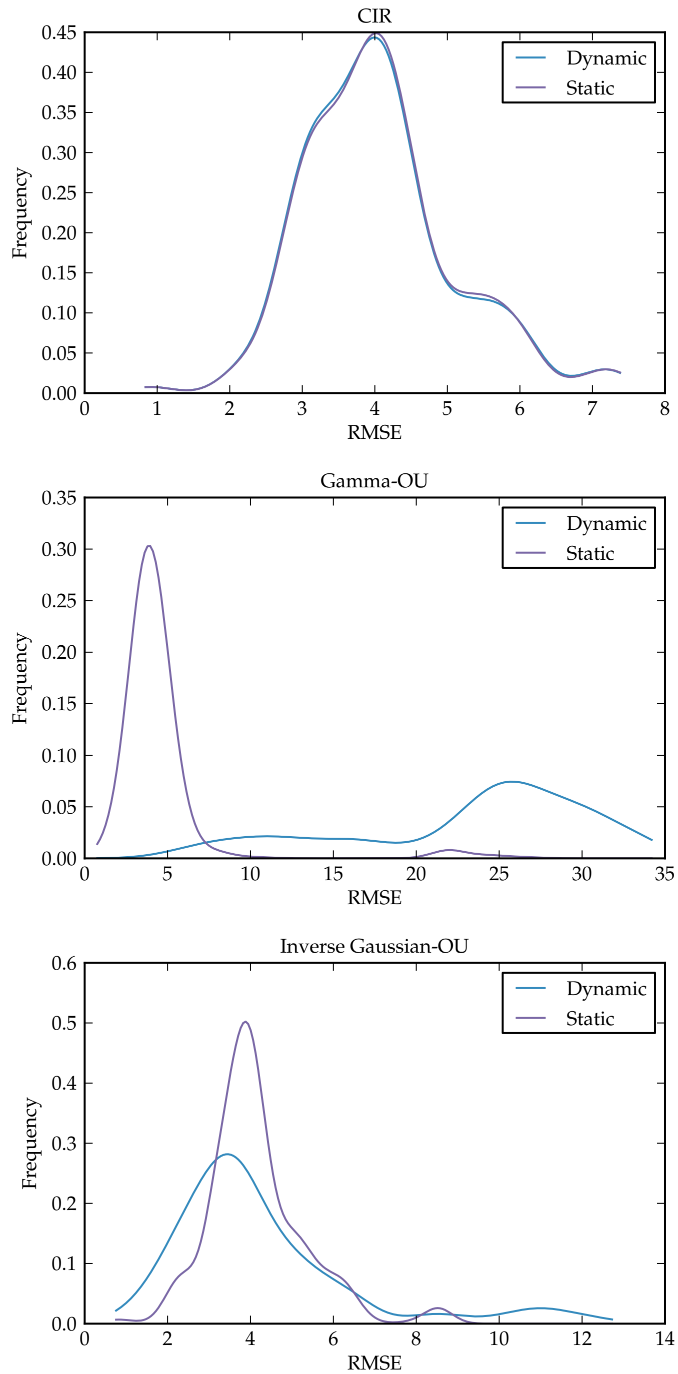

Figure 4.5. Kernel density estimation of the RMSE for the CIR, Gamma-OU, and Inverse Gaussian-OU processes. All models are calibrated to the CDX for 500 consecutive observations.

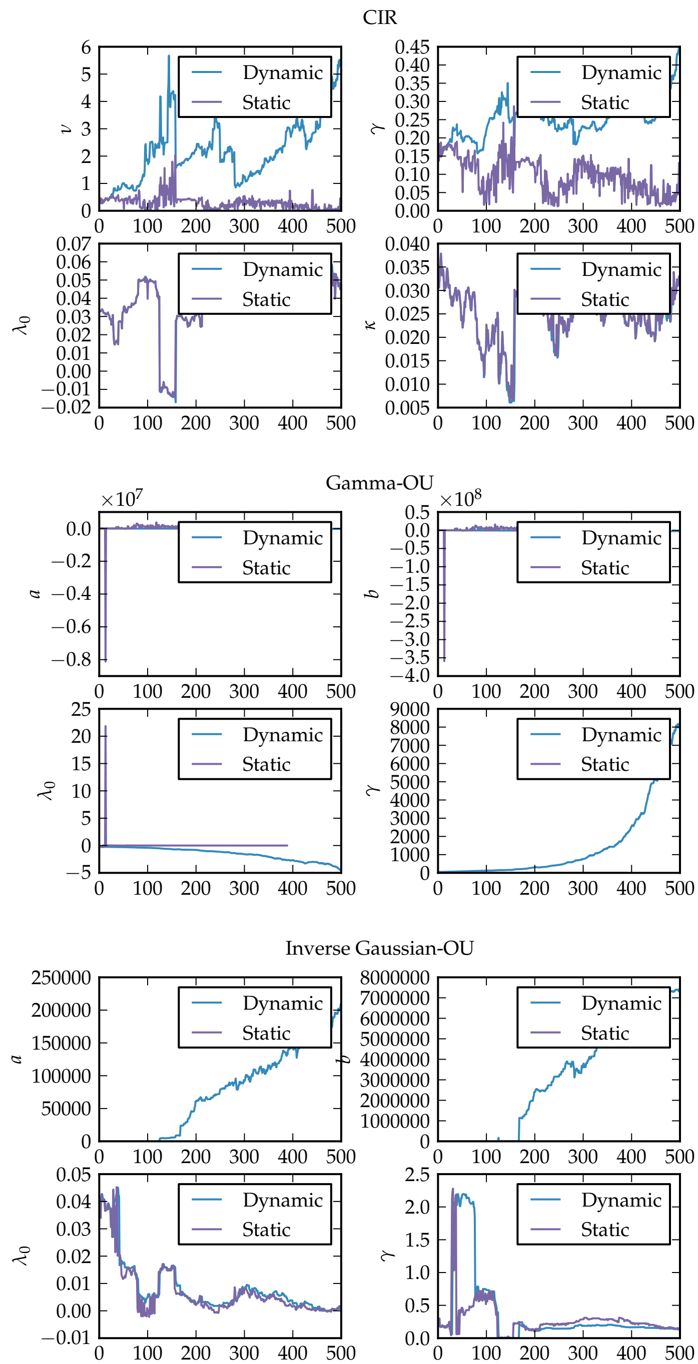

FIGURE 4.6. Stability of the parameters for the CIR, Gamma-OU, and the Inverse Gaussian-OU processes. All models are calibrated to the CDX for 500 consecutive observations.

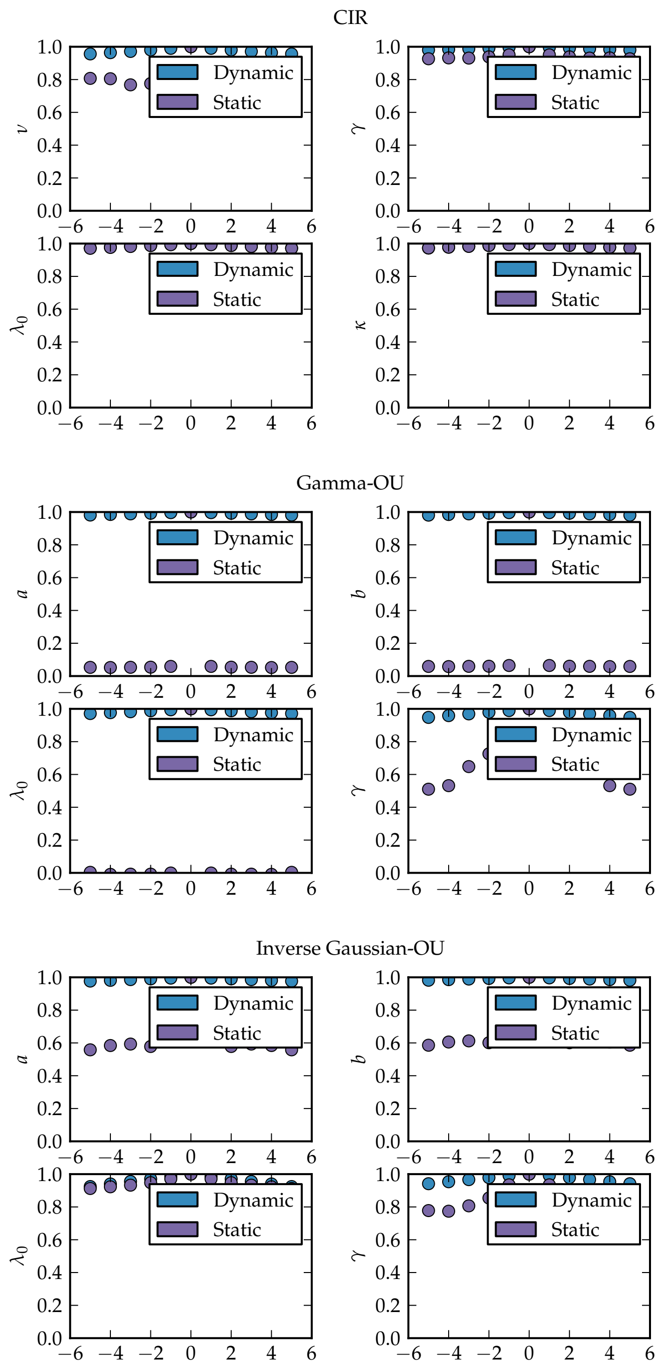

Figure 4.7. Autocorrelograms of the parameters for the CIR, Gamma-OU, and the Inverse Gaussian-OU processes. All models are calibrated to the CDX for 500 consecutive observations.

Chapter 5

Multivariate Intensity Models

We now turn to the modelling of multivariate credit products, with a particular focus on collateralised debt obligations (CDOs). A multivariate credit product is a credit derivative whose payoff is affected by the default of multiple reference entities. This chapter will first describe the most common multivariate credit derivative, the CDO, and proceed to introduce the most common methods of pricing multivariate credit derivatives.

5.1. Introduction to CDOs

A CDO is a complex credit derivative that has been described as playing a pivotal role in the recent housing bubble and the ensuing financial crisis of 2008 [33, 34]. Figure 1.1 illustrates the explosive growth in CDO issuance in the decade leading up to the financial crisis.

A CDO allocates the credit risk of a portfolio of assets to various tranches, with more senior ones preferentially receiving cash-flows.[1] Simplistically, a CDO is a promise to pay cash flows to investors in a prescribed sequence, based on how much cash flow the CDO collects from the pool of bonds or other assets it owns. If cash collected by the CDO is insufficient to pay all of its investors, those in the lower layers (*junior* tranches) suffer the first losses.

There are two major kinds of CDOs - *cash* and *synthetic*. In a cash CDO, a portfolio of assets, such as bonds or loans, are purchased by the CDO manager and cash-flows arising from these assets are distributed to investors. In a synthetic CDO, a synthetic CDO manager sells credit protection via a portfolio of credit default swaps, and cash-flows are derived from derivatives are distributed to investors. Due to greater standardisation and liquidity, we henceforth exclusively consider synthetic CDOs for our purposes.

We now turn to the issue of pricing CDOs and other multivariate credit derivatives. Both CDOs and CDSs are functions of default probability of the reference entities, recovery values, and interest rates. The key distinction between the two credit derivatives is that the value of a CDO is dependent on the correlation structure between the CDSs in the reference portfolio. Accurately modelling this correlation structure is a key open problem in financial mathematics, and various attempts have been proposed in recent years (see [14, 32] for surveys on this topic).

[1] The term *tranche* comes from the French word for slice. By allowing the customisation of risk profiles, tranches have the "ability to create one or more classes of securities whose rating is higher than the average rating of the underlying collateral asset pool or to generate rated securities from a pool of unrated assets" [28].

As in the single-name case, there are two key methods of modelling multivariate credit models: the industry standard latent-variable approach, descending from the structural model of Merton [48], and the intensity approach, principally developed in [22, 24, 56, 59]. We now explain these methods in detail.

5.2. Copulas and Correlation

Throughout the rest of this thesis, we will constantly be seeking to model a dependence structure between random variables. The *copula approach* is a popular method for introducing arbitrary dependence between a set of random variables. We proceed by introducing the definition and basic properties of a copula functions.

Lemma 5.1. *Let X denote a continuous random variable with distribution function F. Then $Z = F(X)$ has a uniform distribution on* $[0, 1]$.

Proof. Let $u \in [0, 1]$. Then

$$\begin{aligned}\mathbb{P}(Z < u) &= \mathbb{P}(F(X) < u) \\ &= \mathbb{P}(X < F^{-1}(u)) \\ &= u\end{aligned}$$

□

Recall that the joint distribution function completely characterises the dependence structure of a sequence of random variables.

Definition 5.2 (Joint distribution function)**.** *The joint distribution function F of the random variables $X_1, X_2, \ldots, X_n$ is*

$$F(\mathbf{x}) = \mathbb{P}\left(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n\right).$$

The basic idea of using copulas in analysing dependency is that the joint distribution function F consists of two separate parts. The first is represented by the distribution functions of the random variables, and the other part is the dependence structure between the random variables which is described by the copula function. We now introduce the definition of the copula.

Definition 5.3 (Copula)**.** *A copula is any function $C : [0, 1]^n \to [0, 1]$ which has the following properties:*

- *(i)* $C(1_{-i}, v_i) = v_i$ *for all* $i = 1, \ldots, n$, $v_i \in [0, 1]$.
- *(ii)* $C(\mathbf{v}) = 0$ *if at least one co-ordinate of the vector v is 0.*
- *(iii) For all* $\mathbf{a}, \mathbf{b} \in [0, 1]^n$, *with* $\mathbf{a} \leq \mathbf{b}$, *the volume of the hypercube with corners a and* $\mathbf{b}$ *is positive, that is.*

$$\sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} C(v_{i_1}, \ldots, v_{i_n}) \geq 0,$$

where $v_{j_1} = a_j$ and $v_{j_2} = b_j$ for all $j = 1, \ldots, n$.

These conditions ensure that a copula defines a distribution function on $[0,1]^n$.

The central theorem of copula theory, providing the basis for the application of copulas to multivariate dependence, is Sklar's theorem.

Theorem 5.4 (Sklar [60]). *Let $X_1, \ldots, X_n$ be random variables with marginal distribution functions $F_1, \ldots F_n$ and joint distribution function F. Then there exists an n-dimensional copula C such that for all $x \in \mathbb{R}^n$,*

$$F(\mathbf{x}) = C\left(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\right)$$

Furthermore, if $F_1, \ldots, F_n$ are absolutely continuous, then C is unique.

Typically, we will consider the case where the marginal distribution functions F_i are continuous and strictly increasing. In this case, the copula of their joint distribution function is given by

$$C(\mathbf{u}) = C(u_1, \ldots, u_n) = F\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right) \tag{5.5}$$

where the functions F_i^{-1} are the inverse marginal distribution functions.

5.3. The Latent-Variable Approach

We now turn to the structural approach to modelling multivariate risk, often referred to as the latent-variable approach. We develop the market-standard one-factor Gaussian model introduced by Li [46], following the exposition in [58].

Consider modelling over a finite time horizon T a portfolio of n obligors. We assume that for each obligor i, we can extract the term structure of survival probabilities $\mathbb{P}^{(i)}_{Surv}(t)$ from the single name market.[2]

The *health* of an obligor is defined to be equal to the latent variable

$$A^{(i)} = \rho_i Z + \sqrt{1-\rho_i^2} Z_i$$

where $Z, Z_i, i = 1, \ldots, n$ are independent standard normal random variables. The coefficient ρ_i determines to correlation of the latent variable with the common factor Y.

Clearly, the vector of latent variables $A^{(i)}$ is multivariate normally distributed with a correlation matrix given by

$$\begin{pmatrix} 1 & \rho_1\rho_2 & \cdots & \rho_1\rho_n \\ \rho_1\rho_2 & 1 & \cdots & \rho_2\rho_n \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1\rho_n & \rho_2\rho_n & \cdots & 1 \end{pmatrix}.$$

Now, for each obligor i, define the barrier function $K_t^{(i)}$ by

$$K_t^{(i)} = \Phi^{-1}(1 - \mathbb{P}^{(i)}_{Surv}(t)).$$

[2]See Table 4.1 and the accompanying discussion for an example of this procedure.

Analogous with the structural approach in §1.2, we finally define the default time for obligor i, $\tau^{(i)}$, by

$$\tau^{(i)} = \inf\{t > 0 \mid A^{(i)} \leq K_i(t)\} = \left(1 - \mathbb{P}^{(i)}_{Surv}\right)^{-1} \left(\Phi\left(A^{(i)}\right)\right)$$

where $K_i(t)$ is our barrier function. Note that we have that the probability for default of the i^{th} obligor is equal to

$$\begin{aligned}
\mathbb{P}\left(A^{(i)} \leq K^{(i)}_t\right) &= \mathbb{P}\left(A^{(i)} \leq \Phi^{-1}\left(1 - \mathbb{P}^{(i)}_{Surv}(t)\right)\right) \\
&= \Phi\left(\Phi^{-1}\left(1 - \mathbb{P}^{(i)}_{Surv}(t)\right)\right) \\
&= 1 - \mathbb{P}^{(i)}_{Surv}(t) \\
&= \mathbb{P}^{(i)}_{Def}(t).
\end{aligned}$$

and so the one-factor model has the correct marginal distributions. By Monte Carlo simulation and the efficient algorithm of [2], this approach allows for the efficient calculation of the distribution of the number of defaults in the portfolio. Appendix A2 shows an example calculation of the loss distribution in the homogenous, constant-correlation case.

If we calibrate a Gaussian copula to market-observed prices of CDO tranches (examined in §6.1), we typically observe a sloped structure similar to that in Figure 5.1. This graph illustrates the market implied constant parameter value of ρ for various 0%-K% CDX tranches at two different points of time. Due to the light tailed behaviour of the normal distribution, senior tranches (tranches with higher attachment levels) have very high implied correlations - up to 98%, reflecting a market-implied tailed dependency that is unaccounted for by the Gaussian copula model [58].

This is a relatively simple model, and has been extended in various directions. Andersen et al. [2] use a Student's t distribution to model the latent variables. Hull and White [36] use the Student's t distribution to model the common factor and the individual term.[3] Andersen and Sidenius [1] introduce stochastic factor weights ρ_i and stochastic recovery rates.

Unfortunately, the one-factor approach has several well-documented drawbacks [58, 59, 62]. Notably, the model is *static*, in that it is only concerned with the number of defaults up to time T, and not the timing of them. Typically, market participants are interested in the time evolution of the portfolio default process, and thus must seek other models. Finally, due to the light tailed behaviour of the normal distribution, the one-factor Gaussian copula model cannot be calibrated to observed market data for reasonable correlation parameters, as it cannot accurately model the observed distribution of joint default probabilities [57]. These difficulties can all be surmounted by a dynamic reduced-form approach to multivariate credit modelling.

[3]Note that these approaches are not equivalent, as the Student's t distribution is not stable.

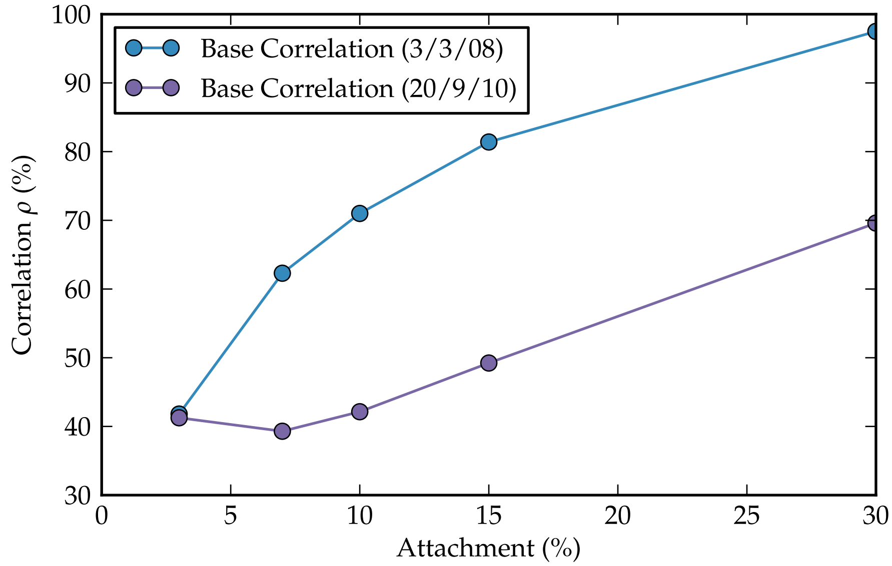

FIGURE 5.1. Market implied Gaussian copula correlations for various 0%-K% CDX tranches during (3/3/08) and after the credit crisis (20/9/10). The parameter ρ is the flat correlation parameter in the Gaussian copula model that recovers the market value of the 0%-K% tranche. Note the higher correlations during the credit crisis, and the artificially high correlation parameters for senior tranches. *Source:* Goldman Sachs.

5.4. Default Intensity Approaches

We now review the major approaches to incorporating default correlation within the intensity model framework of §1.2.

There are three major models that we consider here. The first are the *conditionally independent defaults* models, where the obligors default intensities are correlated by making them dependent on a set of common factors and an idiosyncratic factor. The second approach considered are the *contagion models* of Davis and Lo [22] and Jarrow and Yu [40], where when a firm defaults, the default intensities of related firms jump upwards. The final approach considered are the copula-based methods of Schubert and Schönbucher [59], where the marginal default probabilities are estimated and then transformed into a joint distribution via a copula function.

We now fix notation to be used furthermore. Throughout, we consider $i = 1, \dots, n$ different obligors (typically firms) with default intensities $\lambda_t^{(i)}$ and $\tau^{(i)}$.

5.4.1. Conditionally Independent Default Models. In conditionally independent models, we consider the default intensities $\lambda_t^{(i)}$ to be driven by both state variables X_t and independent idiosyncratic intensities $\overline{\lambda}_t^{(i)}$. As in the latent-factor model, default correlation is specified through the dependence of each obligor on the state variables X_t.

Consider the conditionally independent defaults (CID) model from [23]. The dynamics of this model are given as follows:

$$dr_t = dX_t^1 + dX_t^2 \tag{5.6}$$

$$d\lambda_t^{(i)} = a_{\lambda_i}^1 dX_t^1 + a_{\lambda_i}^2 dX_t^2 + d\overline{\lambda}_t^{(i)} \tag{5.7}$$

$$d\overline{\lambda}_t^{(i)} = \kappa_i \left(\theta_i - \overline{\lambda}_t^{(i)}\right) dt + \sigma_i \sqrt{\overline{\lambda}_t^{(i)}}\, dW_t^{(i)} \tag{5.8}$$

where $\kappa_i, \sigma_i, \theta_i, a_{\lambda_i}^1, a_{\lambda_i}^2$ are constant coefficients and $W_t^{(1)}, \ldots, W_t^{(n)}$ are independent Brownian motions. Notice that the dynamics of the idiosyncratic intensities $\overline{\lambda}_t^{(i)}$ follow a Cox-Ingersoll-Ross process.

The common factors X_t^1, X_t^2 are typically calibrated to the term structure of risk-free rates.

The CID modelling methodology has been challenged in the literature [36, 56]. The principal argument is that the default correlations generated by this procedure are too low compared to empirical results.

5.4.2. Contagion Mechanisms. Contagion models extend the CID models by incorporating the empirical fact that defaults are clustered. That is, defaults tend to concentrate in certain periods of time (default contagion), and the default of one firm can trigger the default of related firms (counterparty contagion).

The Davis and Lo [22] model incorporates default contagion effects into multivariate intensity models. Each firm has an initial intensity rate of $\lambda_t^{(i)}$, which follows any aforementioned intensity processes. When a default occurs, then the default intensity for all remaining firms is increased by a factor $a > 1$, called the *enhancement factor*, to $a\lambda_t^{(i)}$. This augmentation lasts for an exponentially distributed period of time, after which the contagion period ends.

5.4.3. Copula Intensity Models. A novel approach to incorporating correlation in intensity models comes from [56, 59]. This approach, conceptually the intensity counterpart to the structural model of Li [46], differs from the models of the previous sections by not seeking to couple the default intensities $\lambda_t^{(i)}$, and instead couple the *default thresholds* u_i.

Recall that in an intensity setup, the random default time of the i-th obligor $\tau^{(i)}$ is defined by the equation

$$\tau^{(i)} = \inf\left\{ t \geq 0 \mid \exp\left(-\int_0^t \lambda_t^{(i)} \right) \leq u_i \right\}$$

where u_i is a standard uniform random variable independent of $\lambda_t^{(i)}$. In the Schubert and Schönbucher approach, these default thresholds u_i are linked via an arbitrary copula function. Under fairly simple assumptions on the copula C and intensity processes $\lambda_t^{(i)}$, we can derive the following.

Lemma 5.9 (Proposition 4.3 of [59]). *Consider a model where we have n obligors, and consider the i-th obligor for $1 \leq i \leq n$ with associated intensity $\lambda^{(i)}$. Let C be an n-dimensional copula, coupling* *the threshold random variables u_i. Let*

$$\gamma_i(t) = \exp\left(-\int_0^t \lambda_t^{(i)}\right),$$

$$\boldsymbol{\gamma}_{-i}(t) = \left(\gamma_1(t), \ldots, \gamma_{i-1}(t), \gamma_{i+1}(t), \ldots, \gamma_n(t)\right),$$

$$\boldsymbol{\gamma}(t) = \left(\gamma_1(t), \ldots, \gamma_n(t)\right).$$

Let $\mathcal{G}_t$ represent the filtration containing information on $\lambda_u^{(i)}, 0 \le u \le t$ and also the information on the survival time $\{\tau^{(i)} \ge u\}, 0 \le u \le t$.

Then if no obligor has defaulted until time t then

$$\mathbb{P}\left(\tau^{(i)} > T\right) = \mathbb{P}^i_{Surv}(T) = \mathbb{E}\left(\frac{C\left(\boldsymbol{\gamma}_{-i}(t), \gamma_i(T)\right)}{C\left(\boldsymbol{\gamma}(t)\right)} \mid \mathcal{H}_t\right)$$

Furthermore, define an intensity process $h_i(t)$ by the formula[4]

$$\left.\frac{\partial\, \mathbb{P}^i_{Surv}(t, T)}{\partial T}\right|_{T=t} = 1_{\{\tau^i > t\}} h_i(t)$$

Then we can show that

$$h_i(t) = \lambda_t^{(i)} \gamma_i(t) \frac{\frac{\partial}{\partial x_i} C\left(\boldsymbol{\gamma}(t)\right)}{C(\boldsymbol{\gamma}(t))} = \lambda^{(i)}(t) \gamma_i(t) \frac{\partial}{\partial x_i} \ln C(\boldsymbol{\gamma}(t)).$$

The interpretation is that given the additional information that none of the other obligors have defaulted yet, we can infer information about the i-th default threshold u_i, and thus we adjust the default intensity $\lambda_t^{(i)}$ for the i-th obligor to compensate for this. We call $h_i(t)$ the *real* intensities and $\lambda_t^{(i)}$ the *pseudo* intensities.

The dynamics of the real intensity are also affected by the default of other obligors. This is to be expected by the coupling of our default thresholds u_i. Indeed, we have the following theorem.

Lemma 5.10 (Proposition 4.7 of [59]). *If the pseudo default intensities follow diffusion processes, then the dynamics of the default intensity are*

$$dh_i = \frac{C_{x_i}}{C} \cdot \lambda^{(i)} \gamma_i \cdot \left[\left(\frac{d\lambda^{(i)}}{\lambda^{(i)}} - \lambda^{(i)}\, dt\right) - \sum_{j=1}^n \left(\frac{C_{x_i x_j}}{C_{x_i}} - \frac{C_{x_j}}{C}\right) \gamma_j \lambda^{(j)}\, dt\right]$$

if there is no default at time t, and a jump of

$$\Delta h_i = \lambda^{(i)} \gamma_i \frac{C_{x_i}}{C} \left[\frac{C_{x_j x_i}}{C_{x_j}} \frac{C}{C_{x_i}} - 1\right]$$

if the obligor $j \neq i$ defaults at time t. Here, we suppress the dependence on t, $\boldsymbol{\gamma}(t)$ and use the subscript notation for partial derivatives.

[4] In the univariate deterministic intensity case, we can show that $h(t) = \lambda_t$.

Thus the dynamics of the *real* intensities $h_1, \ldots, h_n$ include default contagion effects implied by the default threshold copula. In contrast to the contagion models of [22, 40], contagion effects arise endogenously, rather than through external specification.

Chapter 6

Empirical Analysis of Multivariate Intensity Models

We now seek to examine pricing and sensitivities of various multivariate credit derivatives, using the structural model of §5.3, along with the non-Gaussian Orstein-Uhlenbeck intensity models of §4.2.3. We seek to extend the results of [1, 2, 46] by considering pricing various multivariate credit derivatives using non-Gaussian copula methods with IG-OU and G-OU marginal distributions. We seek to examine properties of model-implied prices, correlation sensitivities, and distribution of extreme numbers of defaults.

6.1. The k-th to Default Basket Swap and CDO Tranches

We now introduce several simplified multivariate credit derivatives, which will be examined in a computational context in later sections. More detailed descriptions of these products are given by [19, 29, 47].

We first consider the pricing of an idealised k-th to default basket swap. This is a multivariate credit derivative specified by choosing n reference entities and the contract tenor T.

The payoff of our idealised k-th to default swap is calculated as follows. Consider a realisation of the default times of our n entities $\tau^{(1)}, \ldots, \tau^{(n)}$, with $\tau^{(1)} \leq \tau^{(2)} \leq \cdots \leq \tau^{(n)}$. Then the derivative pays off $V = 1$ at time $\tau^{(k)}$ if $\tau^{(k)}$ occurs before the contract tenor T. That is, the contract pays off if and only if the k-th default in the basket occurs before time T.

By martingale pricing techniques, the fair price of a k-th to default basket swap is then seen too be equal to

$$V_0 = \mathbb{E}\left(P(0, \tau^{(k)}) 1_{\{\tau^{(k)} < T\}}\right) \tag{6.1}$$

where our discount factor $P(0, \cdot)$ is assumed to be independent of the default times $\tau^{(k)}$.

Note that the price of this derivative is determined by the joint distribution of the default times $(\tau^{(1)}, \ldots, \tau^{(n)})$, and thus modelling the dependency structure of these random variables is critical in accurately determining the price. This contract is valuable as implied prices of the contract allow us to explicitly determine the implied probability of k or more defaults occurring in a given time period.

We also consider the pricing of stylised CDO tranches in our multivariate intensity setup.[1] That is, we consider a contract on a basket of n obligors with attachment points K% and L% and tenor T

[1] This stylised contract differs in several major ways from a traded CDO tranche. In particular, traded CDO exchanges a regular spread payment, as in a CDS, instead of an upfront premium, as in our stylised case

such that the payoff $V_T^{K,L}$ of our contract at time T is given by

$$V_T = \begin{cases} \frac{L}{L-K} & \text{if more than } L\% \text{ of obligors default before time } T \\ \frac{x-K}{L-K} & \text{if } L\% \geq x \geq K\% \text{ of obligors default before time } T \\ 0 & \text{if less than } K\% \text{ of obligors default before time } T \end{cases}$$

As before, the fair price V_0 of a K%-L% CDO tranche is equal to

$$V_0 = \mathbb{E}\left(P(0,T)V_T\right)$$

where our discount factor $P(0,\cdot)$ is assumed to be independent of the default times $\tau^{(k)}$.

6.2. Simulation of the Joint Default Distribution

Following the suggestion of [46], we simulate the joint distribution of default times $\tau^{(i)}$ as follows.

(i) Generate n correlated uniform random variables u_i on $[0,1]$ from an n-dimensional copula.
(ii) Translate the uniform random variables u_i into the default times by inverting the marginal distribution function $\tau^{(i)} = \left(\mathbb{P}_{Def}^{(i)}\right)^{-1}(u_i)$ where $\mathbb{P}_{Def}^{(i)}(t) = \mathbb{P}(\tau^{(i)} \leq t)$.

We consider introduce the three classes of copula models considered in our model. Recall (5.5), where we showed that a copula is determined by a joint distribution function F and n marginal distribution functions F_i by

$$C(\mathbf{u}) = C(u_1,\ldots,u_n) = F\left(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\right).$$

Constructing a copula is then equivalent to specifying the distribution functions F and F_i.

(i) The *Gaussian copula* is defined by letting F being the distribution function of the multivariate normal distribution $N(0,\Sigma)$ and F_i being the distribution function of a standard normal distribution.
(ii) The *Student's t copula* is defined by letting F be the distribution function of the multivariate t distribution $t_\nu(0,\Sigma)$ where Σ is the covariance matrix and ν is the number of degrees of freedom. The marginal distributions F_i are t distributions with ν degrees of freedom. Note that as $\nu \to \infty$, the Student's t copula approaches the Gaussian copula.

Simulation procedures for these copulas are discussed in Appendix A3. Figure 6.1 depicts simulations from Gaussian and Student's t copulas.

6.3. Methodology

We consider the pricing of a k-th to default basket swap on n obligors and various values of k. We assume a flat correlation structure, so that the copula covariance matrix Σ satisfies $(\Sigma)_{ij} = 1$

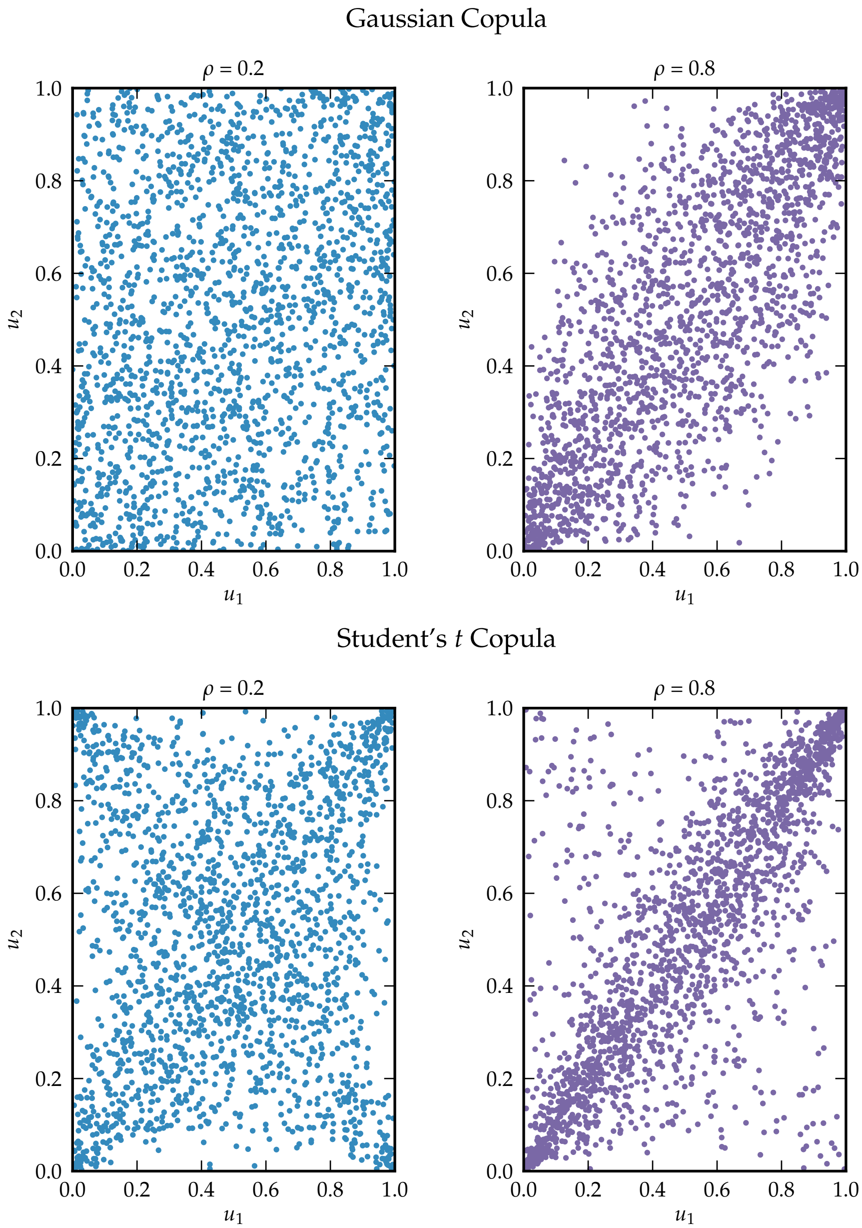

FIGURE 6.1. Simulation of 2000 random draws from bivariate Gaussian and Student's t copulas. The covariance matrix Σ satisfies $(\Sigma)_{ij} = 1$ if $i = j$ and ρ otherwise. The Student's t copula has 2 degrees of freedom. Note the higher tail-dependence of the Student's t copula as compared to the Gaussian copula.

if $i = j$ and ρ otherwise. We assume that the single-name CDS term structure of our obligors is available and identical across obligors.

We price our derivative using Monte Carlo simulation of the default times $\tau^{(i)}$ and (6.1). We examine the sensitivity of option prices to

(i) Various levels of the correlation parameter ρ,

(ii) Type of copula function C,

(iii) Model of the intensity process λ_i.

Formally, we calculate our the price of our k-th to default basket swap by the following approach.

(i) Given an intensity model λ_i and single names CDS spreads, calibrate our intensity model and obtain our model implied default distribution function $\mathbb{P}^{(i)}_{Def}(t)$.

(ii) Simulate N draws of correlated uniform random variables $(u_1, \ldots, u_n)$ from our n-dimensional copula C with correlation parameter ρ.

(iii) Invert the default distribution function $P^{(i)}_{Def}(t)$ to obtain simulated default times $(\tau^{(1)}, \ldots, \tau^{(n)})$.

(iv) Given the simulated default times $\tau^{(i)}$, calculate the value of k-th to default basket swap.

Figure 6.2 illustrates the simulation of default times in a bivariate case with Gamma-OU marginal distributions. Notice that the higher tail dependence of the Student's t copula is reflected in the simulated default times $\tau^{(i)}$.

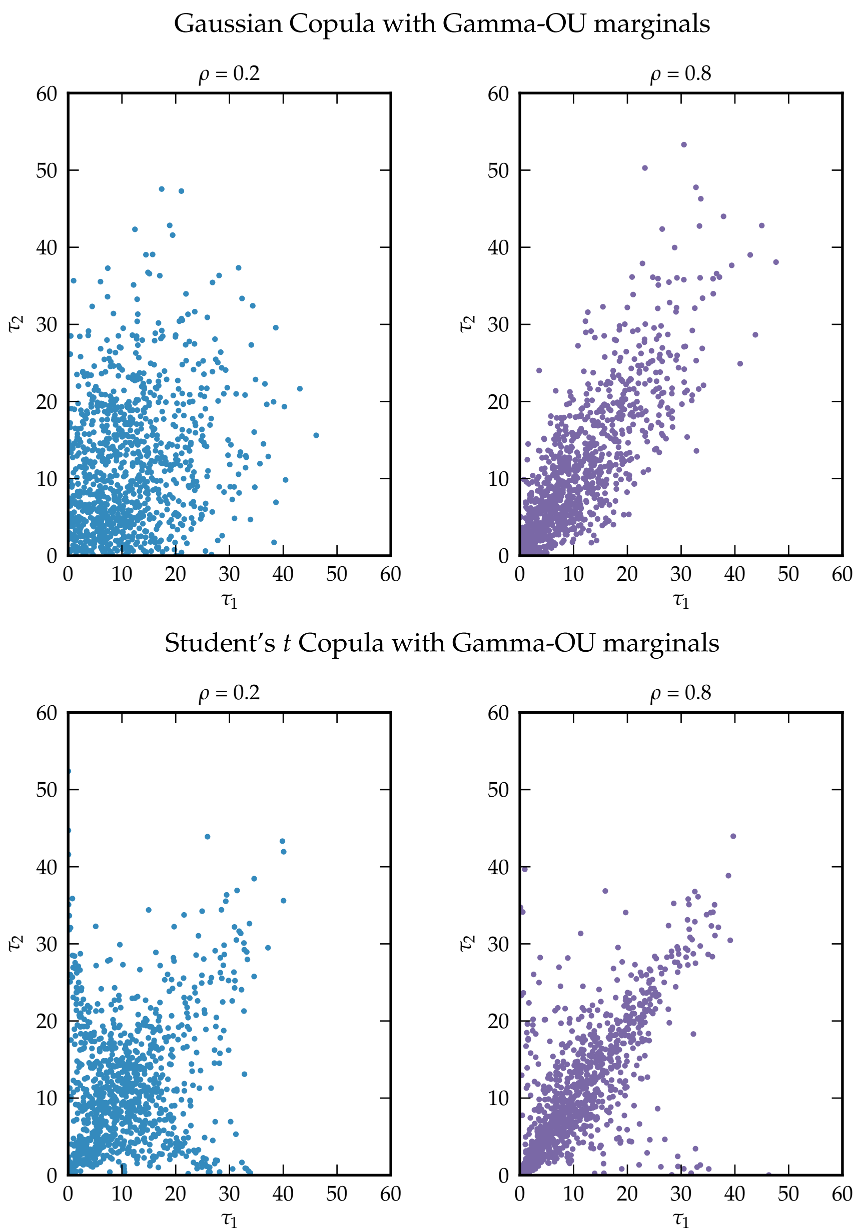

FIGURE 6.2. Simulation of 1000 default times from bivariate Gaussian and Student's t copulas. The covariance matrix Σ satisfies $(\Sigma)_{ij} = 1$ if $i = j$ and ρ otherwise. The Student's t copula has 2 degrees of freedom.

6.4. Computational Results

6.4.1. ***k*-th to Default Basket Swaps.** We now consider the results of the procedure in the previous section. Figure 6.3 contains prices of 1st, 5th, and 10th to default basket swap prices for a range of copulas, intensity models, and correlation parameters.

There are several key conclusions that we can draw from this data. In particular, prices for senior tranches (higher values of k) derived from the Student's t copula are higher than corresponding prices from a Gaussian copula model. This is due to the higher degree of tail dependence in the Student's t distribution, and correspondingly increasing the probability of more extreme joint default events occurring, and thus increasing the payoff of the k-th to default basket swaps.

6.4.2. CDO Tranches. To quantify the impact of copula tail dependence on the pricing of the various contracts, we consider the Value at Risk (VaR) of various stylised CDO tranches. The α% VaR of our financial contract paying off V_T at time T is defined as

$$\sup\left\{c \in \mathbb{R} \mid \mathbb{P}(V_T > c) \leq 1 - \alpha\right\}.$$

That is, an α% VaR for a time period T is the level c such that the probability of our contract paying off more than c is less than α.[2] The VaR is typically estimated by Monte Carlo methods.

Our results are broadly supportive of previous results in the literature [30]. For senior tranches, the Student's t copula generates lower premiums, reflecting the fact that the Student's t distribution has heavier tails than the Gaussian distribution, and thus generates higher probabilities of extreme events. As a corollary, the prices of 1st to default contracts are higher for Gaussian copula models as compared with the Student's t copula models.

Note also that as the correlation parameter ρ tends to 1, k-th to default basket swap prices converge to a constant for all values of k, reflecting the fact that the likelihood of one default is equal to the likelihood of n defaults when defaults are perfectly correlated.

The pricing of CDO tranches is more ambiguous, and indicates that the 5% VaR for various CDO tranches is insensitive to the pricing copula used. This is potentially due to the simulation procedure, as only 5,000 n-dimensional simulations were generated for each value of the correlation parameter ρ. Thus, not enough extreme events were observed in our simulation to distinguish between the Student's t and Gaussian copulas.

As in the univariate case, we see that non-Gaussian OU processes can generate plausible correlation dependency structures for various basket swaps and CDO tranches. In the absence of traded price histories for these structures, we can accept these processes as effective candidates for constructing our multivariate intensity models.

[2]Typically, the VaR is defined for risk management purposes as $\inf\{c \in \mathbb{R} \mid \mathbb{P}(V_T < c) \leq 1 - \alpha\}$; that is, a measure of the probability of *losing* more than c in a given time period. This distinction makes little difference for our purposes.

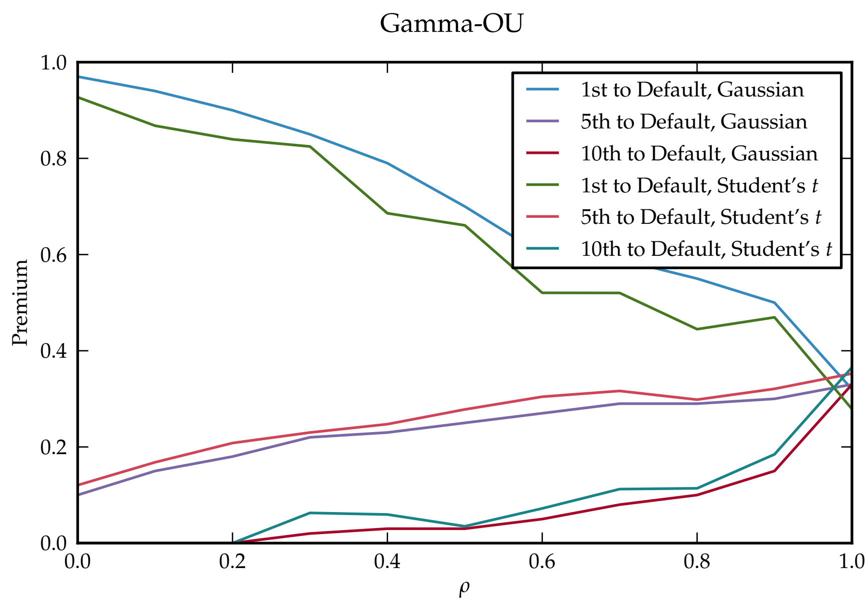

Figure 6.3. Relationship between the price of various k-th to default basket swaps and the flat correlation parameter ρ. We consider a k-th to default basket swap with 10 obligors and calibrated Gamma-OU marginal survival distributions. The Student's t copula has 2 degrees of freedom.

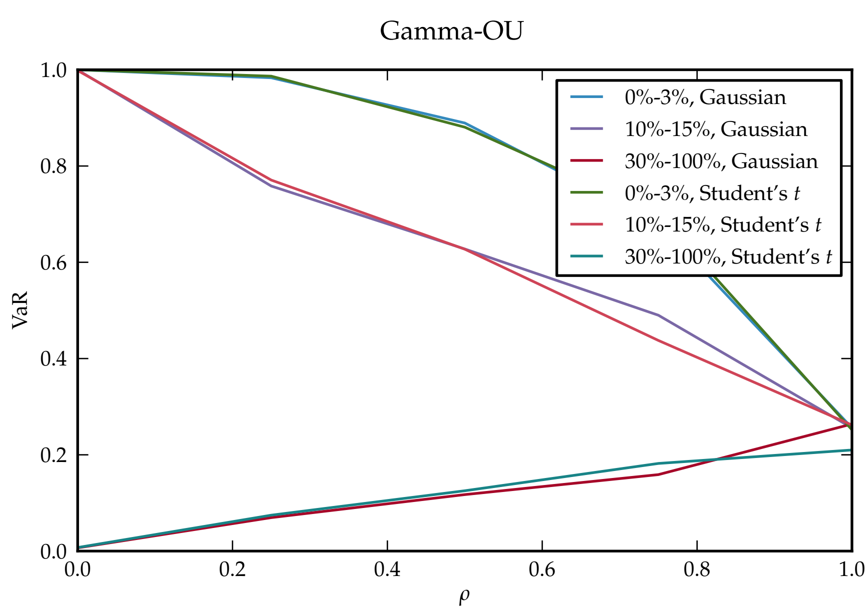

Figure 6.4. Relationship between the 5% VaR of various CDO tranches and the flat correlation parameter ρ. We consider a CDO with a tenor of 5 years containing 100 obligors and calibrated Gamma-OU marginal survival distributions . The Student's t copula has 2 degrees of freedom.

Chapter 7

Summary

7.1. Further Extensions

There has been a vast amount of research conducted in the field of intensity models for credit risk, of which our work examines only a small subset.

In the multivariate context, we have focused on the *bottom-up* approach to credit risk modelling. Thus, we examine individual intensities $\lambda^{(i)}$ and construct the joint distribution of default times through a variety of techniques (§5.4.1, §5.4.2, §5.4.3). This has been shown to be an effective method of incorporating correlation effects in multivariate credit modelling [18]. Unfortunately, the vast number of parameters can be hard to consistently calibrate to market data

An alternative approach, that we have broadly neglected in this work, is the *top-down* approach, where the portfolio loss and number of defaults are modelled directly as the fundamental objects. The top-down approach has become more and more popular, as it provides a much simpler specification in cases where modelling the individual obligors in a portfolio is unnecessary, such as in credit default swap indices [27]. This approach has also been successfully applied in some scenarios [18].

We have also neglected to consider the modelling of the recovery rate R, instead assuming it as a given constant. During the credit crisis, the Gaussian copula model with fixed recovery rates was unable to be calibrated[1] to senior CDO tranches, and in response to this, stochastic recovery rate models were adopted by market participants [43]. First introduced in [1], the recovery rate R is typically modelled as a function of the common market factor, idiosyncratic risk factors, and the default triggering variable. This model allows for consistent calibration and pricing to market data for senior CDO tranches in highly distressed credit environments.

The computational aspects of credit risk modelling have also been a topic of much research. In particular, the topic of efficiently calibrating bottom-up credit risk models for multivariate credit derivatives has been addressed by [19], seeking to resolve some of the computational difficulties with applying simple optimisation algorithms to the high-dimensional nonlinear equations arising in the calibration context. Other research in the multivariate credit risk context has focused upon estimating copulas from observed data [42, 55]. Parameter estimations have been stymied by the lack of data - to quote Davis and Lo [22];

[1]Recall the sample calibration of the Gaussian copula model to the CDX in Figure 5.1 and the accompanying discussion.

> The most obvious feature of this whole area is lack of data. Default events are infrequent and data collected over many years can hardly be supposed to be a sample from a stationary process. Thus reliable statistical estimates of distributional parameters are practically impossible to obtain, particularly the key correlation estimates. In these circumstances there is no point at all in introducing complicated models with lots of additional parameters.

7.2. Conclusions

We have examined univariate and multivariate intensity models for credit risk, with a particular focus on the use of Lévy processes. We first introduce our Lévy processes, and then consider the pricing and calibration of intensity models for single-name credit default swaps. We then assess our various stochastic intensity models by calibrating them to market data and examining properties of these fits.

We then outline the most common multivariate credit risk models, with a particular focus on multivariate intensity approaches. We finally examine the pricing of multivariate credit derivatives in a bottom-up copula model using our non-Gaussian Ornstein-Uhlenbeck Lévy processes for marginal default time distributions.

Throughout, we see that our non-Gaussian Ornstein-Uhlenbeck Lévy models provide empirically appealing term-structure calibrations and sensitivities. These models have several theoretical and empirical advantages over the market-standard deterministic intensity models.

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Appendix A

Further Derivations

A1. A Proof of the Lévy-Itô Decomposition

Recall the Lêy-Itô decomposition (Theorem 2.9), quoted below for convenience.

Theorem A.1 (The Lévy-Itô decomposition). *Let X_t be a Lévy process. Then we can decompose X_t as*

$$X_t = \beta t + \sigma B_t + J_t + M_t$$

where B_t is a Brownian motion, J_t is a compound Poisson process, and M_t is a pure jump martingale.

The final step in the proof required us to show that the expression

$$\underbrace{\int_{|x|<1} \left(e^{iux} - 1 - iux\right)\, dv}_{(\star\star\star)}$$

was the characteristic exponent of a square integrable pure jump martingale that almost surely has a countable number of jumps on a finite interval. We now seek to prove this result. Our proof draws from [44].

Lemma A.2. *Let (ξ_i) be a sequence of independent and identically distributed random variables with distribution function F which does not assign mass to the origin. Let N_t be an independent Poisson process with rate λ. Assume that $\int_{\mathbb{R}} |x|\, dF < \infty$. Then*

(i) The process M_t defined by

$$M_t = \sum_{i=1}^{N_t} \xi_i - \lambda t \int_{\mathbb{R}} x\, dF \tag{A.3}$$

is a martingale with respect to its natural filtration.

(ii) Moreover, if $\int_{\mathbb{R}} x^2\, dF < \infty$ then M is a square integrable martingale such that

$$\mathbb{E}(M_t^2) = \lambda t \int_{\mathbb{R}} x^2\, dF \tag{A.4}$$

Proof.

(i) Note the second term in (A.3) is the compensator of the compound Poisson process $\sum_{i=1}^{N_t} \xi_i$, and so M_t is a martingale by definition.

(ii) By the independence and independent distribution properties, we have

$$\begin{aligned}
\mathbb{E}(M_t^2) &= \mathbb{E}\left(\left(\sum_{i=1}^{N_t}\xi_i\right)^2\right) - \lambda^2 t^2\left(\int_{\mathbb{R}} x\,dF\right)^2 \\
&= \mathbb{E}\left(\sum_{i=1}^{N_t}\xi_i^2\right)^2 + \mathbb{E}\left(\sum_{i\neq j}\xi_i\xi_j\right) - \lambda^2 t^2 (x\,dF)^2 \\
&= \lambda t\int_{\mathbb{R}} x^2\,dF + \mathbb{E}(N_t^2 - N_t)\left(\int_{\mathbb{R}} x\,dF\right)^2 - \lambda^2 t^2\left(\int_{\mathbb{R}} x\,dF\right)^2 \\
&= \lambda t\int_{\mathbb{R}} x^2\,dF + \lambda^2 t^2\left(\int_{\mathbb{R}} x^2\,dF\right)^2 - \lambda^2 t^2\left(\int_{\mathbb{R}} x\,dF\right)^2 \\
&= \lambda t\int_{\mathbb{R}} x^2\,dF
\end{aligned}$$

as required. □

Now, we introduce notation. Let $n = 1, 2, \ldots$. Then let

(i) N_t^n be independent Poisson processes with rate λ_n
(ii) (ξ_i) a sequence of (mutually independent) independent and identically distributed random variables with distribution function F which does not assign mass to the origin and $\int_{\mathbb{R}} x^2\,dF < \infty$.
(iii) M_t^n is the associated martingale, defined in Lemma A.2.
(iv) The filtration $\mathcal{F}_t$, the common complete, right continuous filtration generated by all the processes M^n.

With this in mind, we have the following result.

Lemma A.5. *If*

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}} x^2\,dF_n < \infty \tag{A.6}$$

then there exists a Lévy process X_t which is also a square integrable martingale and whose characteristic exponent is given by

$$\psi(u) = \int_{\mathbb{R}}\left(e^{iux} - 1 - iux\right)\sum_{n\geq 1}\lambda_n\,dF_n. \tag{A.7}$$

Proof. Fix $T > 0$. First, note that $X_t^k = \sum_{n=1}^{k} M_t^n$ is a square integrable martingale. We can also show that by independence of M^n, we have

$$\mathbb{E}\left(\sum_{n=1}^{k} M_t^n\right)^2 = t\sum_{n=1}^{k}\lambda_n\int_{\mathbb{R}} x^2\,dF_n < \infty \tag{A.8}$$

by assumption.

We claim that the sequence $\{X_t^k \mid k \geq 1\}$ is Cauchy in the L_T^2 norm $\|\cdot\|$. Let $k \geq l$. Then

$$\begin{aligned}\|X^k - X^l\|^2 &= \mathbb{E}\left(X_T^k - X_T^l\right)^2 \\ &= T\sum_{n=l}^{k} \lambda_n \int_{\mathbb{R}} x^2\, dF_n \\ &\to 0\end{aligned}$$

as $k, l \to \infty$ by assumption. By the martingale convergence theorem, there then exists a martingale X_t with respect to the filtration $\mathcal{F}_t$. By Doob's martingale inequality, we can show that

$$\lim_{k\to\infty} \mathbb{E}\left(\sup_{0\leq t\leq T}(X_t - X_t^k)^2\right) = 0.$$

Thus the finite dimensional distributions of X_t^k converge to X_t. Consequently,

$$\begin{aligned}\mathbb{E}\left(e^{iu(X_t - X_s)}\right) &= \lim_{k\to\infty} \mathbb{E}\left(e^{iu\left(X_t^k - X_s^k\right)}\right) \\ &= \lim_{k\to\infty} \mathbb{E}\left(e^{iuX_{t-s}^k}\right) \quad \text{as } X_t^k \text{ is a Lévy process} \\ &= \mathbb{E}\left(e^{iuX_{t-s}}\right)\end{aligned}$$

so that X_t has stationary and independent increments.

We also have that the characteristic exponent $\psi(u)$ of X_t is given by

$$\begin{aligned}e^{\psi}(u) &= \lim_{k\to\infty} \prod_{n=1}^{k} \mathbb{E}\left(e^{iuM_1^n}\right) \\ &= \lim_{k\to\infty} \exp\left(\int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) \sum_{n=1}^{k} \lambda_n dF_n\right)\end{aligned}$$

which converges by (A.6).

Right continuity and independence of the time horizon T can be disposed of by a closure result in metric space theory. Thus X_t is a Lévy process with the required properties. $\square$

We are now in a position to prove the Lévy-Itô decomposition result. Note that we have

$$\begin{aligned}\int_{|x|<1} \left(e^{iux} - 1 - iux\right)\, dF_n = \sum_{n=0}^{\infty} \Bigg[\lambda_n \int_{2^{-(n+1)}\leq|x|<2^{-n}} &\left(e^{iux} - 1\right)\, dF_n \\ -iu\lambda_n &\left(\int_{2^{-n+1}\leq|x|<2^{-n}} x dv\right)\Bigg]\end{aligned}$$

where

$$\lambda_n = \nu\left\{x \mid 2^{-(n+1)} \leq |x| < 2^{-n}\right\}$$

and

$$F_n(dx) = \frac{1}{\lambda_n} \nu(dx) \left|_{\left\{2^{-(n+1)} \leq |x| < 2^{-n}\right\}}\right.$$

Then the existence of a square integrable pure jump martingale X_t is given by Lemma A.5. This completes the proof of the Lévy-Itô decomposition.

A2. Loss Distribution in the One-Factor Gaussian Copula

Consider the one-factor Gaussian copula model developed in §5.3. We seek to derive expressions for the distribution of the loss in a synthetic CDO containing n obligors. Assume $\rho_i = \rho$, the flat correlation assumption, and $\mathbb{P}^{(i)}_{Surv}(T) = p$ for all i, the identical default probability. Assume that all obligors have equal recovery rates and nationals.[1]

Recall that the obligor i defaults before time T if the random variable

$$A^{(i)} = \rho Z + \sqrt{1-\rho^2} Z^{(i)}$$

falls below the default threshold $C = \Phi^{-1}(p)$. Thus, conditional on the realisation of the common component Z, the probability of asset i defaulting is

$$\Pi(Z) = \Phi\left(\frac{C - \rho Z}{\sqrt{1-\rho^2}}\right)$$

Thus the conditional loss is the sum of n independent random variables with an expected value of $\Pi(Z)(1-R)N$, where R is the recovery rate and N is the notional. If we assume that the number of issuers is sufficiently large, the law of large numbers applies and thus the conditional loss is exactly $\Pi(Z)(1-R)N$.

Thus under the Gaussian copula model, the probability of the total portfolio loss L being greater than some level K is equal to

$$\mathbb{P}(L > K) = \mathbb{E}\left(1_{\{\Pi(Z)(1-R)N > K\}} \mid Z\right) = \mathbb{P}(Z < B) = \Phi(B)$$

where

$$B = \frac{1}{\rho}\left(C - \sqrt{1-\rho^2}\,\Phi^{-1}\left(\frac{K}{N(1-R)}\right)\right).$$

[1] These assumptions are not necessary, but they simplify the following derivation. This approach is referred to as the large homogenous portfolio (LHP) approximation.

Furthermore, we can calculate $\mathbb{E}\left(\min(L,K)\right)$, an expression used in calculating tranche prices in CDOs. We have

$$\begin{aligned}
\mathbb{E}\left(\min(L,K)\right) &= \mathbb{E}\left(K1_{\{L>K\}} + 1_{\{L\leq K\}}\right) \\
&= K\Phi(B) + \mathbb{E}\left(L1_{\{L<K\}}\right) \\
&= K\Phi(B) + \mathbb{E}\left(\mathbb{E}\left(L1_{\{L\leq K\}} \mid Z\right)\right) \\
&= K\Phi(B) + \mathbb{E}\left(\mathbb{E}\left(\Pi(Z)(1-R)N1_{\{L\leq K\}} \mid Z\right)\right) \\
&= K\Phi(B) + (1-R)N\mathbb{E}\left(\mathbb{E}\left(\Pi(Z)1_{\{Z>B\}} \mid Z\right)\right) \\
&= K\Phi(B) + (1-R)N\int_A^{\infty} \Phi\left(\frac{C-\rho z}{\sqrt{1-\rho^2}}\right)\varphi(z)\,dz \\
&= K\Phi(B) + (1-R)N\Phi_{2,-\rho}(C,-B)
\end{aligned}$$

where $\Phi_{2,-\rho}$ is the bivariate Gaussian distribution function with correlation ρ.

A3. Sampling from Copula Functions

We present several algorithms to sample n uniformly distributed random variables from a variety of copula functions.

Algorithm A3.1 Returns n correlated uniform random variables from a Gaussian copula $C_{\mathbf{\Sigma}}^{Gaussian}$ with covariance matrix $\mathbf{\Sigma}$.

procedure GaussianCopulaSimulation(n, $\mathbf{\Sigma}$)
  Calculate $\mathbf{A}$, the Cholesky decomposition of $\mathbf{\Sigma}$
  Draw an n-dimensional vector $\mathbf{z}$ of independent standard normal variables
  $\mathbf{x} \leftarrow \mathbf{A}z$
  $\mathbf{u} \leftarrow 0$
  for all $x \in \mathbf{x}$ **do**
    $u \leftarrow \Phi(x)$ where $\Phi(\cdot)$ is the cumulative distribution function of the normal distribution
  end for
  Then $\mathbf{u} \sim C_{\mathbf{\Sigma}}^{Gaussian}$
  return u
end procedure

Algorithm A3.2 Returns n correlated uniform random variables from a Student's t copula $C^t_{\mathbf{\Sigma},\nu}$ with covariance matrix $\mathbf{\Sigma}$ and ν degrees of freedom.

procedure StudentsTCopulaSimulation(n, $\mathbf{\Sigma}$, ν)
    Calculate $\mathbf{A}$, the Cholesky decomposition of $\mathbf{\Sigma}$
    Draw an n-dimensional vector $\mathbf{z}$ of independent standard normal variables
    $\mathbf{x} \leftarrow \mathbf{A}z$
    $s \leftarrow$ a random draw from a χ^2_ν distribution
    $y \leftarrow \sqrt{\frac{\nu}{s}}\mathbf{x}$
    $\mathbf{u} \leftarrow 0$
    for all $y \in \mathbf{y}$ **do**
        $y \leftarrow t_\nu(y)$, where $t_\nu(\cdot)$ is the distribution function of the Student's t distribution with ν degrees of freedom
    end for
    Then $\mathbf{u} \sim C^t_{\mathbf{\Sigma},\nu}$
    return u
end procedure

Appendix B

Code Listings

Please note that all code used in this thesis is available at

`https://github.com/zaguar/CDS-Thesis`