MATH 3964 - COMPLEX ANALYSIS

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1. Contour Integration and Cauchy's Theorem

1.1. Analytic functions.

Definition 1.1. A function $f(z)$ is **differentiable** at z_0 if the limit

$$f'(z_0) = \lim_{\zeta \to z_0} \frac{f(\zeta) - f(z_0)}{\zeta - z_0}$$

exists independently of the path of approach.

Definition 1.2. A function $f(z)$ is **analytic** on a region D if it is differentiable everywhere on D. Thus the derivative $f'(z)$ is a function defined on D. a function is analytic at a particular point $z_0 \in \mathbb{C}$ if it is differentiable on some open neighbourhood of z_0.

Theorem 1.3. *A necessary condition for $f(z)$ to be analytic is that if $f(z) = u + iv$, then*

$$v_y = u_x v_x = -u_y$$

Definition 1.4. A function that is analytic throughout the whole complex plain is called an entire function.

Definition 1.5. A point at which a locally analytic function $f(z)$ fails to be analytic is a **singularity** of $f(z)$.

The easiest way to construct analytic functions is as sums of convergent power series. Any power series in the complex domain

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

having positive or infinite radius of convergence R converges to an analytic function in the interior of its disc of convergence, $|z - z_0| < R$.

Theorem 1.6. *Every convergence power series is differentiable term by term in the interior of its disc.*

Proposition 1.7. *The n^{th} derivative of $f(z)$ at $z = z_0$ is*

$$f^{(n)}(z_0) = n! a_n$$

The radius of convergence is given by either of the equivalent exact formulae:

$$R = \liminf_{n \to \infty} |a_n|^{-\frac{1}{n}}$$

or

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$$

Theorem 1.8. *Every power series with a positive or infinite radius of convergence is differentiable term by term to all orders in the interior of its disc of convergence.*

1.2. **Contour integration.** A contour in the complex-plane is just a curve, finite or infinite, which has an arrow or **orientation**. We wish to assign a meaning to the **contour integral**,

$$\int_C f(z)\,dz$$

where C is a contour and $f(z)$ is a function which is defined and piecewise continuous along C

Lemma 1.9 (Triangle inequality for contour integrals)**.**

$$\left|\int_C f(z)\,dz\right| \leq \int_C |f(z)|\,|dz|$$

Lemma 1.10 (The ML formula)**.** *If a contour C has length L and if $|f(z)| \leq M$ on C, then*

$$\left|\int_C f(z)\,dz\right| \leq ML$$

Lemma 1.11 (Jordan's lemma)**.** *Let C_R be all or part of the semicircular contour $Re^{i\theta}$, where θ runs from 0 to π. Suppose that $|f(z)| \leq M(R)$ on C_R and λ is a positive real number. Let*

$$I(R) = \int_{C_R} f(z)e^{i\lambda z}\,dz$$

Then, we have the bound $|I(R)| = \mathcal{O}(M(R))$

1.3. **Cauchy's theorem and extensions.**

Theorem 1.12 (Cauchy's theorem)**.** *If $f(z)$ is analytic on a simply connected region D and if C is any rectifiable closed contour or cycle in D, then*

$$\int_C f(z)\,dz = 0.$$

1.4. **Cauchy's integral formula.**

Theorem 1.13 (Cauchy's integral formula)**.** *Suppose $f(z)$ is analytic in a simply connected region D and that C is a positively oriented rectifiable Jordan curve in D. Then*

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = \begin{cases} f(z_0), & z_0 \in C \\ 0, & z_0 \notin C \end{cases}$$

Theorem 1.14 (Analyticity of Cauchy integrals)**.** *The function*

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

is differentiable to all orders in D and is therefore analytic in D. It's n^{th} derivative is

$$g^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta$$

Theorem 1.15. *Suppose that $f(z)$ is analytic in a simply connected region D. Then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

Corollary. *A function $f(z)$ has an antiderivative in a simply connected region D if and only if $f(z)$ is analytic in D.*

Theorem 1.16 (Removable singularities theorem)**.** *Suppose that $f(z)$ is analytic in a region D except possibly at the point $z_1 \in D$. At z_1, suppose that*

$$\lim_{z \to z_1} (z - z_1) f(z) = 0$$

Then a value of $f(z_1)$ can be assigned so that f(z) becomes analytic at z_1.

Definition 1.17 (Analyticitiy at infinity)**.** Suppose that $f(z)$ is analytic on an unbounded set and let $g(z) = f(1/z)$. Then the point $\infty \in \mathbb{C}^\star$ is point of analyticity of $f(z)$ if $z = 0$ is a point of analyticity of $g(z)$. Similarly, $z = \infty$ is a singularity of a particular type of $f(z)$ if $g(z)$ has a singularity of that same type at $z = 0$.

Definition 1.18. function $f(z)$ which has a singularity at z_0 and is analytic in a deleted neighbourhood of z_0 has a **pole** at z_0, or more specifically, a **pole of order** k, if $(z - z_0)^k f(z)$ is analytic and nonzero at z_0, where k must be a positive integer according to the removable singularities theorem. A pole of order one is a **simple pole**, a pole of order two is a **double pole**, and so on.

Definition 1.19. A function is **meromorphic** if it is analytic in the whole complex plane $\mathbb{C}$ except for poles.

1.5. **The Cauchy-Taylor theorem and analytic continuation.**

Theorem 1.20 (Cauchy-Taylor theorem)**.** *Suppose that $f(z)$ is analytic at z_0 and the disc $D(R) = B(z_0, R)$ is the largest open disc on which $f(z)$ is analytic. Then the Taylor series,*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

converges absolutely to $f(z)$ on $D(R)$ and uniformly on compact subsets.

Theorem 1.21. *Let $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ have a finite radius of convergence R. The radius of convergence of a power series is the distance from the centre to the nearest singularity of its sum function $f(z)$.*

Theorem 1.22 (Cauchy's inequality)**.** *Let $f(z)$ analytic on an open disc $D(\rho)$ with centre z_0. Then, if $|f(z)| \leq M(\rho)$, we have*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M(\rho)}{\rho^n}$$

If a power series $\sum_{k=0}^{\infty} a_k(z-z_0)^k$ converges to $f(z)$, then

$$|a_n| \leq \frac{M(\rho)}{\rho^n}$$

[Liouville's theorem] If an entire function is bounded, or if it possibly grows at a rate such that $f(z)/z \to 0$ uniformly as $z \to \infty$, then $f(z)$ is constant.

Theorem 1.23 (Uniqueness of analytic continuation)**.** *Suppose that $f(z), g(z)$ are analytic in a common region D. Let H be a subset of D that contains a convergent subsequence $\{z_k\}$ whose limit is in the interior of D. If $f(z) = g(z), z \in H$, then $f(z) = g(z)$ everywhere in D.*

1.6. **Laurent's theorem and the residue theorem.**

Theorem 1.24 (Laurent's theorem)**.** *Suppose that $f(z)$ is analytic in the open circular annulus $R_1 \leq |x - x_0| \leq R_2$. Then $f(z)$ admits a power series expansion in both positive and negative powers (a **Laurent series**),*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$$

which is absolutely convergent in D and uniformly convergent on compact subsets.

A formula for the coefficient a_n is

$$a_n = \frac{1}{2\pi i}\int_{C(r)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

Definition 1.25. If z_0 is a pole or isolated essential singularity of $f(z)$, then the **residue** of $f(z)$ at z_0 is the coefficient a_{-1} of $(z-z_0)^{-1}$ in the Laurent expansion of $f(z)$ about z_0. The notation for the residue is $\mathrm{Res}(f, z_0)$.

Theorem 1.26 (Picard's first theorem)**.** *A non-constant entire function has an image either the whole of the complex plane, with at most one exception.*

Lemma 1.27. *If $f(z)$ has a simple pole at z_0, then*

$$Res(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z) = [(z-z_0)f(z)]_{z=z_0}$$

Lemma 1.28. *If $f(z)$ has a pole of order k at z_0, then*

$$Res(f, z_0) = \frac{1}{(k-1)!}\left[\frac{d^{k-1}}{dz^{k-1}}\left((z-z_0)^k f(z)\right)\right]_{z=z_0}$$

Lemma 1.29. *If $f(z)$ and $h(z)$ are analytic at z_0 and $h(z)$ has a simple zero at z_0, then*

$$Res(g/h, z_0) = \frac{g(z_0)}{h'(z_0)}$$

Theorem 1.30 (Residue theorem)**.** *Suppose that $f(z)$ is analytic inside an on a curve C except for a finite number of poles or isolated essential singularities z_i inside C. Then*

$$\int_C f(z)\,dz = 2\pi i \sum_i Res(f, z_i)$$

Definition 1.31 (Residue at infinity)**.** Suppose that $f(z)$ is analytic everywhere outside of a bounded region, in which it admits a convergent Laurent series. Then the residue of $f(z)$ at infinity is minus the coefficient a_{-1}, that is,

$$\mathrm{Res}(f, \infty) = -a_{-1}$$

1.7. **The Gamma function $\Gamma(z)$.**

Definition 1.32. For $\Re(z) > 0$, define

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

The integral converges for $\Re(z) > 0$, and uniformly for $\Re(z) \leq \delta > 0$. Hence $\Gamma(z)$ is analytic at least in the half-plane $\Re(z) > 0$.

Lemma 1.33 (Recurrence relation)**.** *Integration by parts gives $\Gamma(z+1) = z\Gamma(z)$. This gives the analytic continuation to $\Re(z) > -1$. Repeated application provides the analytic continuation to the whole plane except for simple poles at $z = 0, -1, -2, \ldots$.*

Definition 1.34 (Beta function)**.** For $\Re(\alpha), \Re(\beta) > 0$, define the function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

Theorem 1.35. *We have the following relation*

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Lemma 1.36 (Duplication formula)**.**

$$\Gamma(\alpha + \frac{1}{2}) = \frac{\Gamma(2\alpha)\sqrt(\pi)}{2^{2\alpha-1}\Gamma(\alpha)}$$

Lemma 1.37 (Functional relation)**.**

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

1.8. **The residue theorem.**

Lemma 1.38. *In all cases, including isolated essential singularities,*

$$Res(f, z_0)$$

is the coefficient of ϵ^{-1} *in the Laurent expansion*

$$f(z_0 + \epsilon) = \sum_{n \in \mathbb{Z}} a_n \epsilon^n$$

Definition 1.39 (Residue at infinity)**.** Suppose that $f(z)$ is analytic everywhere outside of a bounded region, in which it admits a convergent Laurent series. Then the residue of $f(z)$ at infinity is minus the coefficient a_{-1}, that is,

$$\text{Res}(f, \infty) = -a_{-1}$$

If $f(z) \sim \frac{K}{|z^2|}$ as $|z| \to \infty$, then the residue at ∞ vanishes.

Theorem 1.40 (Argument principle)**.** *Suppose that* $f(z)$ *is analytic on and inside a positively oriented simple closed contour* C*. Suppose also that* $f(z) \neq 0$ *on* C*. Then*

$$\frac{1}{2\pi} \Delta_C \arg f(z) = N - P$$

where N *is the total number of* ***zeroes*** *and* P *is the total number of* ***poles*** *inside* C*, counting multiplicities.*

Example 1.41 (Integration of rational functions)**.** Let $P(x)$ and $Q(x)$ be polynomials with $\deg P(x) = \deg Q(x) - 1$. Then

$$\int_{\mathbb{R}} \frac{P(x)}{Q(x)}\, dx$$

does not exist in the usual sense, but the limit

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx$$

does exist. This limit is called the principle value of the integral.

We then have

$$P \int_{\mathbb{R}} \frac{P(x)}{Q(x)}\, dx = \pi i \,(\text{residue at } \infty) + 2\pi i \sum (\text{residues in the upper half-plane})$$

Example 1.42 (Poles on the real axis)**.** Let $f(x)$ be analytic with a pole on the real axis at $z = x_0$. Then we have

$$P \int_a^b f(x) = \int_{C^+} f(z) dz + \pi i \text{Res}(f, z_0)$$

Example 1.43 (Integrals of trigonometric functions)**.** Let $R(x,y)$ be a rational function bounded on the circle $|z|=1$. Then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{|z|=1} \frac{1}{iz} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\, dz$$

Example 1.44. The integrals

$$\int_{\mathbb{R}} f(x)\cos\alpha x\, dx$$
$$\int_{\mathbb{R}} f(x)\sin\alpha x\, dx$$

are the real and imaginary parts of

$$I = \int_{\mathbb{R}} f(x) e^{i\alpha x}\, dx$$

2. Analytic Theory of Differential Equations

2.1. Existence and uniqueness.

Theorem 2.1 (Existence and uniqueness of first order differential equations)**.** *Let f be an analytic function of two complex variables in the open polydisc $|z-z_0|<a$, $|w-w_0|<b$. The first order differential equation,*

$$\frac{dw}{dz} = f(z,w)$$

has a unique analytic solution $w=w(z)$ such that $w=w_0$ when $z=z_0$ in some disc $|z-z_0|< h, 0<h\le a$

Proof. By repeated differentiation, we can constrict the formal Taylor series

$$w(z) = w_0 + \sum_{n=1}^{\infty} \frac{w^{(n)}(z_0)}{n!}(z-z_0)^n$$

in terms of f and its derivatives, which satisfies the differential equation. We can then show that this series has a positive radius of convergence. □

2.2. Singular point analysis of nonlinear differential equations.

Definition 2.2 (Singular points of differential equations and their solutions)**.** Consider the n^{th}-order differential equation,

$$w^{(n)} = f(z, w, w', w'', \ldots, w^{(n-1)})$$

where f is a locally analytic function of its n complex arguments. A singular point of the differential equation is a point

$$(z_0, w_0, w_0', \ldots, w_0^{(n-1)}) \in \mathbb{C}^n$$

at which f is **not analytic** or a point where one or more of the arguments is **infinite**.

A **fixed** singularity of a differential equation is a singular hyperplane $z=z_0$ or $z=\infty$

Definition 2.3 (Classification of singularities of solutions $w(z)$)**.** Singularities of the solutions are classified as **fixed** or **movable**. A **movable singular point** is a singular point of $w(z)$ depending on one or more integration constants. A **fixed singular point** of $w(z)$ is independent of the integration constants and is included among the fixed singularities of the differential equation.

Example 2.4. (i)

$$w' = w^2$$

with solution $w = -\frac{1}{z-C}$. This has a movable pole at $z = C$.

(ii)

$$w' = \frac{1}{w}$$

with solution $w = \pm\sqrt{2(z-C)}$. This has a moveable quadratic branch point at $z = C$.

(iii)

$$w'' = (w')^2$$

with solution $w = -\log(z - C_1) + C_2$, movable logarithmic branch point at $z = C_1$.

2.3. **Painlevé transcendents.** Consider the differential equation

$$w'' = 6w^2 + g(z)$$

Attempting a Laurent-type expansion about a movable singularity $z = z_0$ leads us to find that the movable terms balance if and only if $p = 2$, $a_0 = 1$. We then find the recurrence relation for a_n. We find that it is of the form

$$(n+1)(n-6)a_n = f_n(a_0, a_1, \dots)$$

and so there is a possible obstruction at $n = 6$ - a **resonance number**. To resolve this, we introduce **logarithmic terms**.

This introduces a **log-pole** at z_0, a logarithmic branch point. To avoid this, we find that we must set $g''(z_0) = 0$, and hence our original differential equation is

$$w'' = 6w^2 + \alpha z + \beta$$

called the Painlevé-I transcendent. When $\alpha = 0$, the system admits the first integral $(w')^2 = 4w^3 + 2\beta w + K$, which is solved with a **Weierstrass elliptic function**, having one double pole in each period parallelogram.

Consider the differential equation

$$w'' = 2w^3 + C(z)w + D(z).$$

Similar analysis yields that $C''(z) = 0$, $D'(z) = 0$. Hence, the differential equation

$$w'' = 2w^3 + (\alpha z + \beta)w + \gamma$$

When $\alpha = 0$, it has the first integral

$$(w')^2 = w^4 + \beta w^2 + 2\gamma w + K$$

which is solved by **Jacobi elliptic functions**. Each period parallelogram has two simple poles, one with residue $a_0 = 1$, the other with $a_0 = -1$. When $\alpha \neq 0$, the DE above defines a new function known as the Painlevé-II transcendent.

2.4. **Fuchsian theory.**

$$w^{(n)} + p_1(z)w^{(n-1)} + \cdots + p_n(z)w = R(z)$$

The **fixed singularities** of the solution $w(z)$ are included among the singularities of the $p_i(z), i = 1, 2, \ldots, n$, and $R(z)$ and possibly $z = \infty$. Linear DE's **cannot have movable singularities**.

Theorem 2.5. *Suppose z_0 is not a singularity of the $p_i(z)$ or $R(z)$. Then the solution of the above linear DE satisfying initial conditions $w^(i) = w^(i)_0$ is analytic in a disc centred at z_0 with radius equal to the distance between z_0 and the nearest singularity of the $p_i(z)$ and $R(z)$.*

Definition 2.6 (Regular singular points)**.** Consider the linear homogenous DE

$$w^{(n)} + p_1(z)w^{(n-1)} + \cdots + p_n(z)w = 0$$

where the $p_i(z)$ are rational functions. The DE has a regular singular point at $t = z_1$ if

- at least one of the p_i has a pole at $z = z_1$
- the **order** of the pole of $p_i(z)$ at $z = z_1$ is at most i for all i.

We have $z = \infty$ is a **regular singular point** if

$$p_i(z) = \mathcal{O}(\frac{1}{z^i})$$

as $z \to \infty$ for all i.

Near a regular singular point at $z = z_1$, one or more particular solutions can be constructed as a power of $z - z_1$ times a convergent power series;

$$(z - z_0)^p \{a_0 + a_1(z - z_0) + \ldots\}$$

The leading powers p satisfy the **indicinal equation**

$$p(p-1)(p-2)\ldots(p-2+1) + q_1 p(p-1)(p-2)\ldots(p-n+2) + q_{n-1}p + q_n = 0$$

where $q_i = \lim_{z \to z_1}(z - z_1)^i p_i(z)$

Definition 2.7 (Fuchsian differential equation)**.** A Fuchsian DE is a linear homogenous DE all of whose singular points are regular.

Hence

- all the $p_i(z)$ are rational functions,
- If z_1 is a pole of any of the p_i then the order of the pole in p_i is less than p_i for all i,
- as $z \to \infty$,
$$p_i(z) = \mathcal{O}(\frac{1}{z^i})$$

Definition 2.8 (Möbius transformations)**.** Fuchsian character is preserved under $\bar{z} = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$. The singular points change position, but their exponents are not affected

If z_1 is a regular singular point, the transformation
$$\bar{w} = (z - z_1)^\lambda w$$
has the following effect:

- all the exponents at z_1 are raised by λ,
- all the exponents at ∞ are lowered by λ,
- exponents are other points are not affected.

Theorem 2.9 (Quadratic transformation)**.** *The transformation $z = \bar{z}^2$ has the following effect:*

- *Exponents at $z = 0$ and $z = \infty$ are doubled.*
- *A singular point at $z + 1 \neq -$ splits into a pair of singular points at $\pm\sqrt{z_1}$ with the* ***same exponents***

2.5. **Hypergeometric functions.** The hypergeometric differential equation
$$z(1-z)w'' + (\gamma - (1 + \alpha + \beta)z)w' - \alpha\beta w = 0$$
is a Fuchsian DE with regular points at 0, 1 and ∞, with exponents

$z = 0$ exponents $0, 1 - \gamma$,
$z = 1$ exponents $0, \gamma - \alpha - \beta$,
$z = \infty$ exponents α, β

The **hypergeometric function** $F(\alpha, \beta; \gamma; z)$ is the particular solution
$$F(\alpha, \beta; \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)}{(\gamma)_n n!} z^n$$
where $(\alpha)_n = \alpha(\alpha + 1)\dots(\alpha + n - 1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$.

The general solution of the hypergeometric equation is
$$y = C_1 F(\alpha, \beta; \gamma, z) + C_2 z^{1-\gamma} F(1 + \alpha - \gamma, 1 + \beta - \gamma; 2 - \gamma; z)$$

We have an integral formula
$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, dt$$

valid for $\Re\gamma > \Re\beta > 0$, $|z| < 1$. but can be extended to $z \in \mathbb{C} = [1, \infty)$.

2.6. **Gauss' formula at $z = 1$.**

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$

2.7. **Kummer's formula at $z = -1$.**

$$F(\alpha, \beta; 1 + \beta - \alpha; -1) = \frac{1}{2}\frac{\Gamma(\frac{1}{2}\beta)\Gamma(1 + \beta - \alpha)}{\Gamma(\beta)\Gamma(1 + \frac{1}{2}\beta - \alpha)}$$

2.8. **Evaluation at $z = \frac{1}{2}$.**

$$F(\alpha, \beta; \gamma; \frac{1}{2}) = 2^{\alpha} F(\alpha, \gamma - \beta; \gamma; -1)$$

$$F(\alpha, \beta; \gamma; \frac{1}{2}) = \frac{\Gamma(\frac{1}{2}\gamma)\Gamma(\frac{1}{2}\gamma + \frac{1}{2})}{\Gamma(\frac{1}{2}\alpha + \frac{1}{2}\gamma)\Gamma(\frac{1}{2} - \frac{1}{2}\alpha + \frac{1}{2}\gamma)}$$

$$F(\alpha, \beta; \frac{1}{2}(\alpha + \beta + 1); \frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2})}{\Gamma(\frac{1}{2}\alpha + \frac{1}{2})\Gamma(\frac{1}{2}\beta + \frac{1}{2})}$$

2.9. **Connection formulae.** We have

$$\begin{aligned} F(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z) &= \\ A\, F(\alpha, \beta; \gamma; z) &+ \\ B\, z^{1-\gamma} F(1 + \alpha - \gamma, 1 + \beta - \gamma; 2 - \gamma; z) & \end{aligned}$$

where

$$A = \frac{\Gamma(1 - \gamma)\Gamma(1 + \alpha + \beta - \gamma)}{\Gamma(1 + \alpha - \gamma)\Gamma(1 + \beta - \gamma)}$$
$$B = \frac{\Gamma(\gamma - 1)\Gamma(1 + \alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}$$

$$\begin{aligned} F(\alpha, \beta; \gamma; z) &= \\ A_1 F(\alpha, \beta; 1 + \alpha + \beta - \gamma; 1 - z) &+ \\ B_1 (1 - z)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; 1 - \alpha - \beta + \gamma; 1 - z) & \end{aligned}$$

where

$$A_1 = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}$$
$$B_1 = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta))}$$

There are various similar formula on page 44 of the notes.

2.10. **Elliptic functions.** There are three main approaches to the study of **elliptic functions**:

- Inversion of elliptic integrals;
- Nonlinear DE's;
- Doubly periodic meromorphic functions.

2.11. **Elliptic integrals.** Consider the class of indefinite integrals,

$$\int R(x, \sqrt{P(x)})\, dx,$$

where R is a rational function of two variables and P is a polynomial without square factors. When $P(x)$ has degree 3 or 4, a Möbius transform of the polynomial $P(x)$ can be given one of several equivalent normalisations:

Jacobi $P(x) = (1 - x^2(1 - k^2x^2))$,
Weierstrass $P(x) = 4x^3 - g_2x - g_3$

Elliptic integral of the **first kind:**

$$F(k, \phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

Elliptic integral of the **second kind:**

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta$$

Elliptic integral of the **third kind:**

$$\Pi(n, k\phi) = \int_0^{\phi} \frac{d\theta}{(1 + n^2 \sin^2 \theta)\sqrt{1 - k^2 \sin^2 \theta}}$$

Complete elliptic integral of the first kind:

$$\begin{aligned} K(k) &= F(k, \frac{\pi}{2}) \\ &= \int_0^1 \frac{dx}{\sqrt{1 - x^2}\sqrt{1 - k^2x^2}} \\ &= \frac{1}{2}\pi F(\frac{1}{2}, \frac{1}{2}, 1; k^2) \end{aligned}$$

Complete elliptic integral of the second kind:

$$\begin{aligned} E(k) &= E(k, \frac{\pi}{2}) \\ &= \int_0^1 \frac{\sqrt{1 - k^2x^2}}{\sqrt{1 - x^2}}\, dx \\ &= \frac{1}{2}\pi F(-\frac{1}{2}, \frac{1}{2}, 1; k^2) \end{aligned}$$

2.12. **Inversion of elliptic integrals.** The **Jacobi elliptic function** $\operatorname{sn} z$ is defined by

$$\int_0^{\operatorname{sn} z} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} = z$$

or equivalently, by the differential equation

$$(w')^2 = (1-w^2)(1-k^2w^2)$$

with $w(0) = 0$ and $w'(0) > 0$.

Then we define $\operatorname{cn} z, \operatorname{dn} z$ by

$$\operatorname{cn} z = \sqrt{1 - \operatorname{sn}^2 z}$$
$$\operatorname{dn} z = \sqrt{1 - k^2 \operatorname{sn}^2 z}$$

2.13. **Doubly periodic meromorphic functions.** We define

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{m,n\in Z \\ m,n\neq 0}} \left(\frac{1}{z - mw - nw')^2} - \frac{1}{(mw+nw')^2} \right)$$

where w and w' are nonzero complex numbers with w'/w not real. It is easy to see that $\wp(z)$ is globally meromorphic with periods w and w'.

If f is any elliptic function, then $\int_C f(z) = 0$, where C is the period parallelogram, as the integrals on opposite sides cancel. Hence, the sum of the residues of all the poles in a period parallelogram.

We have

$$\wp'(z) = -2 \sum_{m,n\in\mathbb{Z}} \frac{1}{(z - mw - nw')^3}$$

Consider the **Laurent expansion** of $\wp$ and $\wp'$ about $z = 0$. We have

$$\frac{1}{(z - mw - nw')^2} = \sum_{k=0}^{\infty} \frac{(k+1)z^k}{(mw - nw')k + 2}$$

Defining $I_{2k} = \sum' \frac{1}{(mw+nw')^{2k}}$ gives

$$\wp(z) = \frac{1}{z^2} + 3I_4 z^2 + 5I_6 z^4 + \dots$$

$$\wp'(z) = -\frac{2}{z^3} + 6I_4 z + 20I_6 z^3$$

where the radius of convergence is the minimum of $|w|$ and $|w'|$.

Letting $g_2 = 60I_4$, $g_3 = 140I_6$, we obtain the DE

$$(w')^2 = 4w^3 - g_2 w - g_3$$

with general solution $\wp(z - z_0)$.

We also have

$$\int^{\wp(z)} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = z$$

2.14. **Jacobi elliptic functions.** We have

$$\mathrm{sn}\,' z = \mathrm{cn}\, z \mathrm{dn}\, z$$

$$\mathrm{cn}\,' z = -\mathrm{sn}\, z \mathrm{dn}\, z$$

$$\mathrm{dn}\,' z = -k^2 \mathrm{sn}\, z \mathrm{cn}\, z$$

deduced from $\mathrm{sn}\, z$ satisfying $w' = \sqrt{1-w^2}\sqrt{1-k^2w^2}$.

For identities of the Jacobi elliptic functions, see pages 57-67 in the notes.

2.15. **Addition theorems.** See pages 57-67 in the notes

2.16. **Liouville theory.** Let an elliptic function be defined as any **doubly periodic meromorphic function**.

Definition 2.10 (Order)**.** The order of an elliptic function is the number of poles of $f(z)$ inside a period parallelogram, counting multiplicities (a pole of order n is counted as n poles.) Then $\wp, \mathrm{sn}\,, \mathrm{cn}\,$ are elliptic functions of order 2.

Theorem 2.11. *An elliptic function of order zero is constant.*

Theorem 2.12. *The sum of the residues of $f(z)$ at all poles in a period parallelogram is zero.*

Theorem 2.13. *The transcendental equation $f(z) = a$ where $f(z)$ is an elliptic function of order m has exactly m roots in every period parallelogram, counting multiplicities, for every $a \in \mathbb{C}$.*

Theorem 2.14. *A Möbius transformation leaves the order and periods of an elliptic function unchanged.*

Theorem 2.15. *Any solution of $(w')^2 = aw^4 + bw^3 + cw^2 + dw + e$ with a, b both not zero, no square factors, is an elliptic function of order 2.*

2.17. **Weierstrass' Theorem.** Let $f(z)$ be an elliptic function of any order. Then it has a unique representation

$$f(z) = R_1(\wp(z)) + R_2(\wp(z))\wp'(z)$$

where $\wp(z)$ is the Weierstrass function having the same periods and R_i rational functions.